Elementary embeddings in torsion-free hyperbolic groups

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Abstract

We consider embeddings in a torsion-free hyperbolic group which are elementary in the sense of first-order logic. We give a description of these embeddings in terms of Sela's hyperbolic towers. We deduce as a corollary that subgroups elementarily embedded in finitely generated free groups are free factors.

1 Introduction

Tarski's problem asks whether any two finitely generated non-abelian free groups are elementary equivalent, namely whether they satisfy the same closed formulas over the language of groups. In a series of articles starting with [Sel01] and culminating in [Sel06], Sela answered this question positively (see also the work of Kharlampovich and Myasnikov [KM06]). Sela's approach is very geometric, and thus enables him in [Sel] to tackle problems on the first-order theory of torsion-free hyperbolic groups as well.

Another notion of interest in first-order theory is that of elementary subgroup, or elementary embedding. Informally, a subgroup H of a group G is elementary if any tuple of elements of H satisfies the same first-order properties in H and in G (see Section 2 for a definition).

Denote by $\mathbb{F}_n$ the free group on n generators. To prove that finitely generated free groups have the same elementary theory, Sela shows in fact the following stronger result:

Theorem 1.1: *[Sel06, Theorem 4] Suppose $2 \leq k \leq n$. The standard embedding $\mathbb{F}_k \hookrightarrow \mathbb{F}_n$ is elementary.*

In this paper, we use some of Sela's techniques to give a description of elementary subgroups of torsion-free hyperbolic groups. The main result is given by

Theorem 1.2: *Let G be a torsion-free hyperbolic group. Let $H \hookrightarrow G$ be an elementary embedding. Then G is a hyperbolic tower based on H.*

Hyperbolic towers are built by successive addition of hyperbolic floors, which can be described as follows. A group G has a hyperbolic floor structure over a subgroup G' if it is the fundamental group of a complex X built by gluing some hyperbolic surfaces $\Sigma_1, \ldots, \Sigma_m$ along their boundary to a disjoint union of complexes $X'_1, \ldots, X'_l$; such that $G' = \pi_1(X'_1) * \ldots * \pi_1(X'_l)$. We require moreover the existence of a retraction $r : G \to G'$ which sends the fundamental groups $\pi_1(\Sigma_j)$ to non-abelian images.

A hyperbolic tower over H is built by successively adding hyperbolic floors to a 'ground floor' which is the free product of H with some free group and some closed surface groups (see Figure 1). For a precise definition, see Definition 5.2.

Hyperbolic towers are defined by Sela in [Sel01], and enable him to give in [Sel06] a description of finitely generated groups which are elementary equivalent to free groups. This structure is also used in [Sel] to give a classification of elementary equivalence classes of torsion-free hyperbolic groups.

In the particular case where G is a free group, we show that Theorem 1.2 implies the converse of Theorem 1.1, so that we have

Theorem 1.3: *Let $n \geq 2$, and let H be a subgroup of $\mathbb{F}_n$. The embedding of H in $\mathbb{F}_n$ is elementary if and only if H is a non-abelian free factor of $\mathbb{F}_n$.*

In the particular case where G is the fundamental group of a closed hyperbolic surface, we show that applying Theorem 1.2 gives

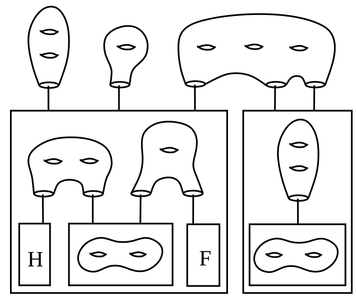

Figure 1: A hyperbolic tower over H.

Theorem 1.4: *Let S be the fundamental group of a closed surface Σ whose Euler characteristic $\chi(\Sigma)$ is at most -1. Suppose H is a proper subgroup of S whose embedding in S is elementary.*

Then H is a non-abelian free factor of the fundamental group of a subsurface Σ_0 of Σ whose complement in Σ is connected, and which satisfies $\chi(\Sigma_0) \geq \chi(\Sigma)/2$ (with equality if and only if Σ is the double of Σ_0).

Thus in the example represented on Figure 2, the fundamental group of the surface Σ_1 is not an elementary subgroup of $\pi_1(\Sigma)$. Note that this can also be proved by remarking that the element corresponding to γ can be written as a product of two commutators in $\pi_1(\Sigma)$, though not in $\pi_1(\Sigma_1)$.

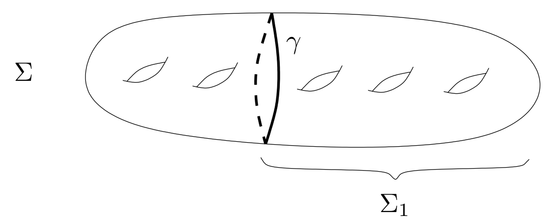

Figure 2: $\pi_1(\Sigma_1)$ is not elementarily embedded in $\pi_1(\Sigma)$.

Section 2 recalls the definition of an elementary embedding. Section 3 gives some basic but key results about surfaces with boundary and maps between such surfaces. Section 4 recalls known results about the existence of factor sets for homomorphisms from a finitely generated group to a torsion-free hyperbolic group, and gives an outline of the proof. In Section 5, we prove Theorem 1.2, 1.3 and 1.4. To prove Theorem 1.2, we express statements derived from the existence of factors sets as first-order formulas satisfied by H. The interpretation of these formulas on G gives us some maps with specific properties that we call preretractions. To complete the proof, we then need another result, Proposition 5.11, which says that the existence of a preretraction guarantees the existence of a hyperbolic floor structure. The last three sections of the paper are essentially devoted to the proof of Proposition 5.11.

Most of the results presented here are part of my Ph.D. thesis. I wish to thank Zlil Sela for suggesting this problem as well as for the many discussions that followed, Gilbert Levitt for his constant assistance, and Thomas Delzant, Panos Papasoglu and Vincent Guirardel for their helpful advice.

2 Elementary embeddings

We give only an informal definition of first-order formulas, for a precise definition and more detailed background, the reader is referred to [Cha], or to [CK90].

The language of groups $\mathcal{L}$ is the set of symbols consisting of

- the symbols $\cdot$, $^{-1}$, and 1, standing for multiplication, inverse and identity element respectively (these are specific to the language of groups);
- the usual first-order symbols: $=$, $\wedge$ (meaning "and"), $\vee$ (meaning "or"), $\neg$ (meaning "not"), $\Rightarrow$, and the quantifiers $\forall$ and $\exists$;
- variables $x_1, x_2, \ldots$

A first-order formula in the language of groups is a finite formula using these symbols. A formula ϕ with no free variables (i.e. all of whose variables are bound by quantifiers) is called a closed formula, and we say that it is satisfied by a group G if the interpretation of the formula is true on G (this is denoted by $G \models \phi$). Note that variables always stand for elements of the group, so that in a first-order formula, we can only quantify on elements of the group and not on integers, say, or on subsets of the group.

Let H be a subgroup of a group G. We can add to the language $\mathcal{L}$ a constant symbol $\lceil h \rceil$ for each element h of H, we denote this extended language $\mathcal{L}_H$.

Definition 2.1: *We say that the embedding of H in G is elementary (or that H is an elementary subgroup of G) if for any closed first-order formula ϕ in the language $\mathcal{L}_H$ we have*

$$H \models \phi \iff G \models \phi.$$

This is denoted as $H \preceq G$.

Example 2.2: Suppose H is an elementary subgroup of some group G. By considering the following closed formula:

$$\phi_h : \forall x \; x\lceil h \rceil = \lceil h \rceil x,$$

we see that an element h of H is in the centre of H if and only if it is in the centre of G.

3 Some preliminary results on surface groups

All the surfaces we consider are, unless otherwise stated, compact and connected, and with (possibly empty) boundary.

Let Σ be a surface, and denote by S its fundamental group. To each boundary component of Σ corresponds a conjugacy class of maximal cyclic subgroups of S: we call these subgroups *maximal boundary subgroups*, and their generators *maximal boundary elements*. A non-trivial element in a maximal boundary subgroup is called a *boundary element*, and the cyclic group it generates is called a *boundary subgroup*.

The group S endowed with the set of its maximal boundary subgroups, is called a *surface group*. If Σ and Σ' are surfaces, a morphism f between their fundamental groups S and S' is a *morphism of surface groups* if it sends boundary elements to boundary elements.

We define a notion of complexity for surfaces with non-empty boundary. We will denote by $\mathrm{rk}(F)$ the rank of a finitely generated free group F.

Definition 3.1: (topological complexity) *Let Σ be a surface with non-empty boundary, denote by S its fundamental group. The topological complexity $k(\Sigma)$ of Σ is the pair $(\mathrm{rk}(S), -n)$, where n is the number of boundary components of Σ. We order topological complexities by the lexicographic order.*

3.1 Surface groups acting on simplicial trees

Definition 3.2: (splitting $\Delta(\Sigma, \mathcal{C})$ dual to a set of simple closed curves) *Let Σ be a surface, and let $\mathcal{C}$ be a set of non null-homotopic, two-sided, disjoint simple closed curves on Σ (note that we do not exclude pairs of parallel curves). We denote $\Delta(\Sigma, \mathcal{C})$ the splitting of the fundamental group S of Σ dual to the set of curves $\mathcal{C}$ given by the Van Kampen Lemma. We call the corresponding S-tree $T_{\mathcal{C}}$ the tree dual to $\mathcal{C}$.*

Theorem III.2.6 of [MS84] shows that if the fundamental group of a surface Σ acts minimally on a simplicial tree T, in such a way that boundary elements are elliptic and edge stabilisers are cyclic, then there exists a set $\mathcal{C}$ of non null-homotopic, two-sided and non boundary parallel simple closed curves on Σ such that T is isomorphic to $T_{\mathcal{C}}$ (as an S-tree). The proof consists in building an equivariant map between a universal cover of Σ and T, which we choose so that the inverse image of midpoints of edges of T give us lifts of non null-homotopic simple closed curves on Σ.

We will give in Lemma 3.5 a slightly different version of this result, which is proved in essentially the same way. One difference is that we do not assume that the edge stabilisers are cyclic, so we get a surjective equivariant simplicial map $T_{\mathcal{C}} \to T$ which is not necessarily injective. Also, we restrict ourselves to sets $\mathcal{C}$ of simple closed curves which are pairwise non-parallel, so that we loose simpliciality of the map $T_{\mathcal{C}} \to T$. Finally, we want to ensure that the map $T_{\mathcal{C}} \to T$ is locally minimal in the following sense:

Definition 3.3: (locally minimal map) *Let G be a group which acts on (the topological realisations of) simplicial trees T and T'. A continuous equivariant map $t : T \to T'$ is said to be locally minimal if it is either constant or locally injective on the interior of each edge, if it sends vertices on vertices, and if for any vertex v of T for which some open neighbourhood of v has image contained in an edge e' of T', the stabiliser $G_{e'}$ of e' in T' is strictly contained in the stabiliser G_v of v in T.*

Note that if the stabiliser of each vertex of T is elliptic in the action of G on T', then we can find a locally minimal equivariant map $t : T \to T'$.

Definition 3.4: (essential set of curves on a surface) *A set $\mathcal{C}$ of simple closed curves on a surface is essential if its elements are non null-homotopic, two-sided, non-boundary parallel and pairwise non-parallel. We consider such sets up to homotopy.*

Lemma 3.5: *Suppose that the fundamental group S of a surface with boundary Σ acts on a simplicial tree T, in such a way that boundary subgroups are elliptic. Then there exists an essential set of curves $\mathcal{C}$ on Σ, and a locally minimal equivariant map $t : T_{\mathcal{C}} \to T$ between the S-tree dual to $\mathcal{C}$ and T.*

Remark 3.6: *The cyclic subgroups of S corresponding to curves in $\mathcal{C}$ stabilise edges of T. The fundamental groups of connected components of the complement of $\mathcal{C}$ in Σ are vertex groups of $\Delta(\Sigma, \mathcal{C})$, thus they are elliptic in T.*

3.2 Maps between surfaces

Let Σ and Σ' be surfaces with non-empty boundary, and let $\phi : \Sigma \to \Sigma'$ be a continuous map which sends $\partial\Sigma$ into $\partial\Sigma'$. We are interested in the corresponding map ϕ_* between the fundamental groups S and S' of Σ and Σ'. It is a morphism of surface groups.

We will now give two results which give sufficient geometric conditions on ϕ to guarantee respectively the injectivity and the virtual surjectivity of the morphism ϕ_*. The first of these results is Theorem 3.1 of [Gab85].

Definition 3.7: (simple arc) *An arc on a surface Σ is a continuous map $a : [0, 1] \to \Sigma$ such that $a(0)$ and $a(1)$ lie in the boundary of Σ. If a is injective, we say it is a simple arc.*

Theorem 3.8: *Let Σ and Σ' be connected surfaces with non-empty boundary, and denote by S and S' their respective fundamental groups. Let $\phi : \Sigma \to \Sigma'$ be a continuous map such that $\phi(\partial\Sigma) \subseteq \partial\Sigma'$. If ϕ does not send any non boundary-parallel simple arc α in Σ to a boundary-parallel arc $\phi(\alpha)$ in Σ', then the corresponding map ϕ_* from S to S' is injective.*

Thus injectivity is guaranteed as soon as non boundary-parallel simple arcs are sent to non boundary-parallel images. The next result guarantees virtual surjectivity of the map ϕ_* as soon as non null-homotopic simple closed curves are sent by ϕ to non null-homotopic images. We give

Definition 3.9: (non-pinching map) *Let Σ be a surface, denote by S its fundamental group. A morphism f from S to a group G is said to be non-pinching with respect to Σ if its kernel does not contain any element corresponding to a non null-homotopic simple closed curve lying on Σ.*

We now have

Lemma 3.10: *Let S and S' be fundamental groups of surfaces with non-empty boundary Σ and Σ' respectively. Let $f : S \to S'$ be a morphism of surface groups. If f is non-pinching with respect to Σ, and if $f(S)$ is not contained in a boundary subgroup of S', then $f(S)$ has finite index in S'.*

To prove it, we will use

Lemma 3.11: *Let Q be the fundamental group of a surface Ξ with non-empty boundary. If Q_0 is a finitely generated infinite index subgroup of Q, it is of the form*

$$Q_0 = C_1 * \ldots * C_m * F$$

where F is a (possibly trivial) free group, $m \geq 0$, each of the groups C_j is a boundary subgroup of Q, and any boundary element of Q contained in Q_0 can be conjugated in one of the groups C_j by an element of Q_0.

Proof. By Theorem 2.1 in [Sco78], there exists a finite covering $p : \Xi_1 \to \Xi$, and a subsurface Ξ_0 of Ξ_1, such that Q_0 is the image by the injection p_* of the fundamental group of Ξ_0. Let $Q_1 = \pi_1(\Xi_1)$, and identify Q_1 to its isomorphic image by p_*. The covering is finite, so Ξ_1 is compact, Q_1 is of finite index in Q, and the boundary elements of Q_1 are exactly the boundary elements of Q contained in Q_1. Since Q_0 is of infinite index in Q, it must be of infinite index in Q_1. Thus Ξ_0 is a proper subsurface of Ξ_1, and at least one of its boundary components is not a boundary component of Ξ_1. In particular, there is a basis of Q_0 as a free group which contains a maximal boundary element corresponding to every boundary component of Ξ_0 which is also a boundary component of Ξ_1. This basis gives the required free factor decomposition of Q_0. □

We can now prove Lemma 3.10.

Proof. Suppose $f(S)$ has infinite index in S'. Then it admits a free product decomposition $C_1 * \ldots * C_m * F$ as given by Lemma 3.11, and $m \geq 1$ since boundary elements of S are sent to boundary elements of S'. If $f(S)$ is not contained in a boundary subgroup of S', this decomposition contains at least two factors, so the corresponding minimal $f(S)$-tree T_0 with trivial edge stabilisers is not reduced to a point. The group S acts via f on T_0, the tree T_0 is minimal for this action, and boundary subgroups of S are sent to boundary subgroups of S', thus they lie in conjugates of the factors C_i and are elliptic in T_0. By Lemma 3.5, we get a set of simple closed curves on Σ whose corresponding elements stabilise edges of T_0 via f, i.e. have trivial image by f. This contradicts the fact that f is non-pinching. □

In the setting of Lemma 3.10, we can also deduce that the complexity of Σ must be greater than that of Σ' thanks to the following lemma.

Lemma 3.12: *Let S and S' be the fundamental groups of surfaces Σ and Σ' with non-empty boundary. If $f : S \to S'$ is a morphism of surface groups such that $f(S)$ is a subgroup of finite index of S', then*

$$k(\Sigma) \geq k(\Sigma');$$

and if we have equality, f is bijective.

Proof. A subgroup of finite index in a finitely generated free group of rank r is a free group of rank at least r, with equality if and only if the index is 1. Thus $\mathrm{rk}(S') \leq \mathrm{rk}(f(S))$ with equality if and only if f is surjective. Now $\mathrm{rk}(f(S)) \leq \mathrm{rk}(S)$, and since free groups are Hopfian, we have equality if and only if f is injective. Thus $\mathrm{rk}(S') \leq \mathrm{rk}(S)$, with equality if and only if f is bijective. If this is the case, f sends non-conjugate boundary subgroups of S to non-conjugate boundary subgroups of S', so that Σ' has at least as many boundary component as Σ, and $k(\Sigma) \geq k(\Sigma')$. □

4 Factors sets

In this section, we recall the result obtained in [Sel] of the existence of a factor set for non-injective homomorphisms into a torsion-free hyperbolic group Γ. We then indicate how to get a relative version (the 'restricted' version of Sela).

4.1 Modular groups

The following definition will be of use.

Definition 4.1: (graph of groups with surfaces) *A graph of groups with surfaces is a graph of groups Λ together with a subset V_S of $V(\Lambda)$ such that any vertex v in V_S satisfies:*

- *there are no loops at v, i.e. no edges both of whose endpoints are v;*
- *there exists a compact connected surface with boundary Σ which is not a disk, a Möbius band or a cylinder, such that the vertex group G_v is the fundamental group S of Σ;*
- *for each edge e adjacent to v, the injection $i_e : G_e \hookrightarrow G_v$ maps G_e onto a maximal boundary subgroup of S;*
- *this induces a bijection between the set of edges adjacent to v and the set of conjugacy classes in S of maximal boundary subgroups of S;*

The vertices of V_S are called surface type vertices. A vertex of the tree T_Λ corresponding to Λ whose projection to Λ is of surface type is also said to be of surface type. The surfaces corresponding to surface type vertices of Λ are called the surfaces of Λ.

Suppose Λ is a graph of groups with surfaces. Denote by G its fundamental group. We are interested in automorphisms of G which preserve in some sense the graph of group decomposition Λ. The group formed by these automorphisms is called the modular group of Λ.

Here is an example of such an automorphism.

Definition 4.2: (Dehn twist) *Suppose a group G has a graph of groups decomposition Λ, and let C be the group corresponding to an edge e in this decomposition. This edge induces a one-edge splitting of G over C. A Dehn twist of this one-edge splitting by an element of the centre of C is called a Dehn twist of Λ about e.*

If ϕ_v is an automorphism of a vertex group G_v of Λ which restricts on each adjacent edge group G_e to conjugation by an element g_e, it can be naturally extended to an automorphism of G, as explained in Section 2.3 of [Lev05]. Such an extension also preserves the graph of group decomposition.

We give in particular

Definition 4.3: (surface type automorphism) *Let Λ be a graph of groups with surfaces, let G denote its fundamental group, and let S be a surface type vertex group in this decomposition. If ϕ_S is an automorphism of S which restricts to conjugation on each maximal boundary subgroup, an extension of ϕ_S to G is called a surface type automorphism of Λ.*

We say that a group G is freely indecomposable if it does not admit any non-trivial free product decompositions of the form $G = G' * G''$. We will only use modular groups in the case where the fundamental group G is torsion-free hyperbolic and freely indecomposable. In this case, the modular group of Λ is generated by the automorphisms we just defined, together with inner automorphisms.

Definition 4.4: (modular group $\mathrm{Mod}(\Lambda)$ of a graph of groups Λ) *Let G be a freely indecomposable torsion-free hyperbolic group. Let Λ be a splitting of G as a graph of group with surfaces. The modular group $\mathrm{Mod}(\Lambda)$ of Λ is the subgroup of $\mathrm{Aut}(G)$ generated by inner automorphisms, Dehn twists, and surface type automorphisms.*

Definition 4.5: (modular group $\mathrm{Mod}(G)$ of a group G) *Let G be a freely indecomposable torsion-free hyperbolic group. We define the modular group of G, denoted by $\mathrm{Mod}(G)$, to be the subgroup of $\mathrm{Aut}(G)$ generated by the modular groups of all the cyclic splittings Λ of G as a graph of groups with surfaces.*

4.2 Factor sets for morphisms to a torsion-free hyperbolic group

We have the following result, which is implied by [Sel, Theorem 1.26]. It can be proved directly by an argument similar to that of [Sel, Theorem 1.25].

Proposition 4.6: *Let Γ be a torsion-free hyperbolic group. Let G be a non-cyclic freely indecomposable hyperbolic group. There exists a finite set of proper quotients of G such that for any non-injective morphism $f : G \to \Gamma$, there is an element σ of $\mathrm{Mod}(G)$ such that $f \circ \sigma$ factors through one of the corresponding quotient maps.*

Such a set of proper quotients is called a factor set for non-injective morphisms $G \to \Gamma$. We give an outline of the proof, which follows that of [Sel, Theorem 1.25]. It is based on the powerful but technical shortening argument.

Definition 4.7: (stable sequence, stable kernel) *Let G be a finitely generated group, and let $(h_n)_{n\in\mathbb{N}}$ be a sequence of morphisms from G to a group G'. The sequence $(h_n)_{n\in\mathbb{N}}$ is stable if for any element g of G, either all but finitely many of the $h_n(g)$ are trivial, or all but finitely many of the $h_n(g)$ are non-trivial. The set of elements g for which the former holds is a subgroup of G that we call the stable kernel of the sequence $(h_n)_{n\in\mathbb{N}}$.*

Note that by a diagonal argument, one can extract a stable subsequence from any sequence of morphisms. For the rest of this section, let Γ be a torsion free hyperbolic group endowed with a finite generating set $D(\Gamma)$.

Definition 4.8: (Γ-limit group) *A Γ-limit group is the quotient of a finitely generated group G by the stable kernel of a stable sequence of morphisms $h_n : G \to \Gamma$.*

Although Γ-limit groups are not necessarily finitely presented if Γ is not free, Sela shows in [Sel, Theorem 1.17]

Theorem 4.9: *If $\eta : G \to L$ is a Γ-limit quotient of G corresponding to a stable sequence h_n of morphisms $G \to \Gamma$, all but finitely many of the maps h_n factor through η.*

Theorem 4.9 also implies the following result, which will be of use later.

Theorem 4.10: *If $(L_i)_{i\in\mathbb{N}}$ is a sequence of Γ-limit groups such that there exist surjective maps $\eta_i : L_i \to L_{i+1}$ for all i, then all but finitely many of the maps η_i are isomorphisms.*

Definition 4.11: (short morphism) *Let G be a group endowed with a finite generating set $D(G)$. A morphism $h : G \to \Gamma$ is said to be short if*

$$\max_{g\in D(G)} |h(g)|_{D(\Gamma)} \leq \max_{g\in D(G)} |\gamma h(\sigma(g))\gamma^{-1}|_{D(\Gamma)}$$

for any element σ of $\mathrm{Mod}(G)$ and γ of Γ. Here $|.|_{D(\Gamma)}$ denotes the word metric in Γ with respect to $D(\Gamma)$.

Definition 4.12: (Γ shortening quotient, maximal Γ shortening quotient) *A Γ shortening quotient Q of a finitely generated group G is the quotient of G by the stable kernel of a sequence of non-injective short morphisms $h_n : G \to \Gamma$.*

We order Γ shortening quotients of a finitely generated group G by the following relation: if Q_1, Q_2 are Γ shortening quotients of G with corresponding quotient maps $\eta_i : G \to Q_i$, we say $Q_1 \geq Q_2$ if there exists a morphism $\tau : Q_1 \to Q_2$ such that $\eta_2 = \tau \circ \eta_1$. A maximal Γ shortening quotient of G is a shortening quotient which is maximal for this order.

Sela shows, using Theorem 4.9, that every Γ shortening quotient of G is smaller than a maximal Γ shortening quotient, and that there are only finitely many maximal Γ shortening quotients $\eta_i : G \to M_i$ (Propositions 1.20 and 1.21 of [Sel]).

Now suppose $f : G \to \Gamma$ is a non-injective morphism, and let σ and γ be elements of $\mathrm{Mod}(G)$ and Γ respectively, such that $h = \mathrm{Conj}(\gamma) \circ f \circ \sigma$ is short. The sequence $(h_n)_{n\in\mathbb{N}}$ of constant term $h_n = h$ is a sequence of non-injective short morphisms, so the quotient of G by its stable kernel (which is just the kernel of h) is a Γ shortening quotient. Thus it is smaller than one of the Γ maximal shortening

quotients M_i, which means that $f \circ \sigma$ factors through the corresponding quotient map η_i. To complete the proof of Proposition 4.6, there remains to show that these Γ maximal shortening quotients are proper.

Proposition 4.13: *If G is a non-cyclic and freely indecomposable hyperbolic group, then Γ shortening quotients of G are proper quotients.*

Theorem 1.25 of [Sel] claims that this holds for G a freely indecomposable Γ limit group. This is what allows Sela to then build Makanin-Razborov diagrams. The main difference is that to show Theorem 1.25 of [Sel], Sela has to deal also with axial components in the limit tree: when G is assumed to be hyperbolic, there are no such components. However, since abelian subgroups of Γ-limit groups are well behaved, the shortening argument can be extended.

Outline of the proof. Let D be a finite generating set for G. Suppose we have a stable sequence $h_n : G \to \Gamma$ of non-injective morphisms whose stable kernel is trivial. This implies in particular that G is torsion-free. Our aim is to show that the h_n cannot be short.

For each n, consider the action of G on the Cayley graph X of Γ 'twisted' by the morphism h_n. Pick a point x_n in X which minimizes the displacement function $x \mapsto \max_{g \in D} d_X(x, h_n(g) \cdot x)$ of this action. Rescale the distance on X by the minimal displacement $\mu[h_n] = \max_{g \in D} d_X(x_n, h_n(g) \cdot x_n)$. We get a sequence (X_n, x_n) of pointed G-spaces.

Thanks to this rescaling, the sequence of actions $(\lambda_n)_{n \in \mathbb{N}}$ converges to an action λ of G on a pointed metric space. This action is non-trivial by choice of basepoints.

The fact that the h_n are non-injective, and that their stable kernel is trivial, implies that they belong to infinitely many conjugacy classes. Thus, up to extraction of a subsequence, the rescaling constant $\mu[h_n]$ tends to infinity.

Now if X is δ-hyperbolic, each X_n is a $\delta/\mu[h_n]$-hyperbolic space, so the limit is a connected 0-hyperbolic space, i.e. a real tree.

Using the fact that G is torsion-free hyperbolic, and that the sequence of hyperbolicity constant of the spaces X_n tends to 0, we can show that the limit action satisfies some nice conditions, such as abelianity of arc stabilisers and triviality of tripod stabilisers.

These conditions allow us to analyse the limit tree with Rips theory (see [Sel97], or [Gui08]), and this gives us a decomposition of G as the fundamental group of a graph of group with surfaces Λ whose edge groups are abelian (and thus cyclic since G is torsion-free hyperbolic). Note that there are no Levitt components since G is freely indecomposable, and no axial components since G is hyperbolic. We can thus use the shortening argument, developed by Rips and Sela in [RS94]: it shows that for any n large enough, we can find an element σ_n of $\mathrm{Mod}(\Lambda)$ such that the action λ_n twisted by σ_n is strictly shorter than λ_n, i.e. the displacement of the basepoint by $\lambda_n \circ \sigma_n$ is smaller than by λ_n. By our choice of basepoint, this implies that the morphisms h_n were not short. □

Note that the non-injectivity of the maps h_n is only used to show that the rescaling constant tends to infinity. Suppose now we are given an infinite sequence of pairwise non-conjugate short injective maps i_n from G to Γ. The stable kernel of such a sequence is trivial. We build (X_n, x_n) as above. The non-conjugacy of the maps i_n is sufficient to ensure that the rescaling constant tends to infinity, so that by following the argument above, we get a contradiction to the shortness of the maps i_n. Thus no such sequence exist, i.e. there is a finite number of conjugacy classes of short embeddings $G \to \Gamma$. This can be formulated by

Theorem 4.14: *Let Γ be a torsion-free hyperbolic group. Let G be a non-cyclic freely indecomposable hyperbolic group. There exists a finite set $\{i_1, \ldots, i_k\}$ of embeddings $G \hookrightarrow \Gamma$ such that for any embedding $i : G \hookrightarrow \Gamma$, there is an index j with $1 \leq j \leq l$, an element γ of Γ, and an element σ of $\mathrm{Mod}(G)$ such that*

$$i = \mathrm{Conj}\,(\gamma) \circ i_j \circ \sigma.$$

4.3 Relative factor sets

One of the most important hypothesis in Proposition 4.13 is the fact that G is freely indecomposable: this is required to show that the limit tree has no Levitt components, a condition which is absolutely essential to make the shortening argument work. But in fact, the absence of Levitt component is also guaranteed if G is only freely indecomposable relative to a subgroup H, provided H is elliptic in the limit tree. To ensure this, we fix an embedding $H \hookrightarrow \Gamma$, and replace a few definitions and arguments by their relative versions.

We say that a group G is freely indecomposable with respect to a subgroup H if it does not admit any non trivial free product decomposition of the form $G = G' * G''$, where H is contained in G'. We start by giving

Definition 4.15: (relative modular group $\mathrm{Mod}_H(G)$) *Let G be a torsion-free hyperbolic group, and let H be a subgroup of G with respect to which G is freely indecomposable. Let Λ be a cyclic splitting of G as a graph of groups with surfaces for which H lies in a non surface type vertex group. The modular group $\mathrm{Mod}_H(\Lambda)$ of Λ relative to H is the subgroup of $\mathrm{Aut}_H(G)$ (the group of automorphisms of G fixing H) generated by the inner automorphisms, Dehn twists, and surface type automorphisms of Λ which restrict to the identity on H.*

The modular group of G relative to H is the subgroup of $\mathrm{Aut}(G)$ generated by the subgroups $\mathrm{Mod}_H(\Lambda)$, where Λ is a cyclic splitting of G in which H lies in a non surface type vertex group. We denote it $\mathrm{Mod}_H(G)$.

In the relative case, the factor set existence result we get is given by

Proposition 4.16: *Let G be a hyperbolic group which is freely indecomposable with respect to a non-abelian subgroup H. Let Γ be a torsion-free hyperbolic group endowed with a fixed embedding $j : H \hookrightarrow \Gamma$. There exists a finite set of proper quotients of G, and a finite subset H_0 of H, such that for any non-injective morphism $h : G \to \Gamma$ which coincides with j on H_0, there is an element σ of $\mathrm{Mod}_H(G)$ such that $h \circ \sigma$ factors through one of the corresponding quotient maps.*

The proof of this proposition is similar to the non-relative case, we will thus only outline the differences. For the rest of this section, let G be a hyperbolic group which is freely indecomposable with respect to a non-abelian subgroup H, fix D a finite generating set for G. Let Γ be a torsion-free hyperbolic group endowed with a fixed embedding $j : H \hookrightarrow \Gamma$ and with a finite generating set $D(\Gamma)$. We will say that a morphism $G \to \Gamma$ fixes H if it coincides with j on H.

The notion of shortness of a morphism $G \to \Gamma$ is now changed to

Definition 4.17: (short morphism relative to H) *A morphism $h : G \to \Gamma$ is said to be short relative to H if*

$$\max_{g \in D} |h(g)|_{D(\Gamma)} \leq \max_{g \in D} |h(\sigma(g))|_{D(\Gamma)}$$

for any element σ of $\mathrm{Mod}_H(G)$.

The difference with the previous case is that we got rid of the conjugation by an element of G: this is because we will only be interested in maps which fix H.

We will also need the following definition.

Definition 4.18: (fixing H in the limit) *Denote by $B_G(r)$ the set of elements of G represented by words in D whose length is at most r. We say that a sequence of morphisms $h_n : G \to \Gamma$ fixes H in the limit if for any r, for all n large enough, the map h_n coincides on $B_G(r) \cap H$ with the fixed embedding $j : H \hookrightarrow \Gamma$.*

Definition 4.19: (Γ shortening quotient relative to H) *A Γ shortening quotient of G relative to H is the quotient of G by the stable kernel of a stable sequence of non-injective morphisms $h_n : G \to \Gamma$ which are short relative to H and fix H in the limit.*

Note that Γ shortening quotients relative to H are in particular Γ-limit groups. Thus, they satisfy the strong descending chain condition given by Proposition 4.9, which is required to prove that every Γ

shortening quotient relative to H is under a maximal such quotient, and that maximal such quotients are in finite number. As before, there only remains to show

Proposition 4.20: *Let G be a hyperbolic group which is freely indecomposable with respect to a non-abelian subgroup H. Let Γ be a torsion-free hyperbolic group endowed with a fixed embedding $j : H \hookrightarrow \Gamma$. Then Γ shortening quotients of G relative to H are proper quotients.*

Outline of the proof. Suppose we have a stable sequence $h_n : G \to \Gamma$ of non-injective morphisms which fix H in the limit, and whose stable kernel is trivial. We want to see that the h_n are not all short relative to H. The sequence $(h_n)_{n\in\mathbb{N}}$ gives a sequence of actions of G on the Cayley graph X of Γ. As before, we choose basepoints x_n, however the choice of basepoints is different: here we take x_n to be simply the vertex corresponding to the identity element of Γ, and we rescale the metric on X by the displacement of the basepoint which is now $\max_{g\in D} |h_n(g)|_{D(\Gamma)}$. We get in the limit an action λ on a pointed metric space (X_n, x_n).

This change in the choice of basepoints matches the change in our definition of shortness of a morphism, and ensures that the implication: 'if the h_n are short relative to H, the actions λ_n thus obtained are short' holds. However, this change also means that the non-triviality of the limit action is not immediate anymore: we only get that the basepoint x of the limit metric space is not a global fixed point.

Again the non-injectivity of the maps h_n implies that they belong to an infinity of conjugacy classes, so that the rescaling constant tends to infinity. Thus λ is an action on a pointed real tree (T, x), and since the morphisms h_n fix H in the limit, H fixes x in the action λ.

As in the non-relative case, the limit G-tree satisfies some nice conditions: in particular its arc stabilisers are abelian. Now if λ has a global fixed point y, it must be distinct from x, but then H stabilises both x and y so it stabilises the arc between them. This contradicts the non-abelianity of H, and we deduce that the limit action is non-trivial.

We analyse the action λ with Rips theory, this gives a decomposition for G as a graph of group with surfaces Λ whose edge groups are cyclic. Since H fixes a point in λ and G is freely indecomposable with respect to H, there are no Levitt components, and since G is hyperbolic, there are no axial components. Thus, the shortening argument gives us elements σ_n of $\mathrm{Mod}(\Lambda)$ to shorten all but finitely many of the actions λ_n, and we can ensure that the σ_n restrict to the identity on H. Since H is elliptic in Λ, the maps σ_n are in fact elements of $\mathrm{Mod}_H(G)$. Thus at most finitely many of the morphisms h_n are short relative to H. ☐

If we start with a sequence of pairwise distinct injective maps i_n from G to Γ which fix H, and are short relative to H, the rescaling constant $\max_{g\in D} |h_n(g)|_{n\in\mathbb{N}}$ still tends to infinity. Thus we can apply a similar argument, and we get a contradiction: this means that there are only finitely many such maps. We get

Theorem 4.21: *Let G be a hyperbolic group which is freely indecomposable with respect to a non-abelian subgroup H. Let Γ be a torsion-free hyperbolic group endowed with a fixed embedding $H \hookrightarrow \Gamma$. There exists a finite set $i_1, \ldots, i_k$ of embeddings $G \hookrightarrow \Gamma$ such that for any embedding $i : G \hookrightarrow \Gamma$ which fixes H, there is an index j with $1 \leq j \leq l$, and an element σ of $\mathrm{Mod}_H(G)$ such that*

$$i = i_j \circ \sigma.$$

4.4 Relative co-Hopf properties

From Theorem 4.21, we can deduce a relative co-Hopf property for torsion-free hyperbolic groups:

Proposition 4.22: *Let G be a torsion-free hyperbolic group. Let H be a non-cyclic subgroup of G relative to which G is freely indecomposable. If $\phi : G \to G$ is injective and fixes H then it is an isomorphism.*

Proof. Suppose ϕ is a strict embedding: then the injective morphisms $\phi^n : G \to G$ all fix H, and their images are are pairwise distinct since they are strictly embedded one into the other: this contradicts Theorem 4.21. ☐

Now we can actually get a stronger statement by using the following lemma, suggested by Vincent Guirardel.

Lemma 4.23: *If a finitely generated group G is freely indecomposable relative to a subgroup H, then H has a finitely generated subgroup H_0 relative to which G is freely indecomposable.*

Proof. Suppose G' is a subgroup of G. Denote by $T(G')$ the set of all simplicial G-trees τ with trivial edge stabilisers in which G' fixes a vertex v_τ. Define

$$A(G') = \bigcap_{\tau \in T(G')} \mathrm{Stab}(v_\tau)$$

To each τ in $T(G')$, we associate the corresponding free product decomposition of G. Since G is finitely generated, the number of factors of such a decomposition is bounded: let $m_G(G')$ be the maximal number of factors that such a decomposition can have. A decomposition with $m_G(G')$ factors is clearly of the form

$$A * B_1 * \ldots * B_r$$

where $B_1, \ldots, B_r$ are freely indecomposable (possibly cyclic), and A contains G' and is freely indecomposable with respect to G'. Such a decomposition corresponds to a tree τ in $T(G')$ so $A(G') \leq A$. But in any tree τ of $T(G')$, A fixes the vertex v_τ, so $A = A(G')$.

If $G' \leq G''$, we have $T(G') \supseteq T(G'')$, so that $A(G') \leq A(G'')$ and $m_G(G') \geq m_G(G'')$, and if we have equality, a maximal decomposition with respect to G'' is also a maximal decomposition with respect to G' so that $A(G') = A(G'')$.

We can now prove the lemma. Let $\{h_1, h_2, \ldots\}$ be a generating set for H, and let $H_k = \langle h_1, \ldots h_k \rangle$ of H. The sequence $(m_G(H_k))_{k>0}$ is non-increasing, so it must stabilise, thus the sequence $A(H_k)$ stabilises after some index k_0. In particular $H_k \leq A(H_k) \leq A(H_{k_0})$ for all k, so $H \leq A(H_{k_0})$. But $A(H_{k_0})$ is a free factor of G: since we assumed G freely indecomposable with respect to H, we must have $A(H_{k_0}) = G$, and G is freely indecomposable with respect to H_{k_0}. □

We get a partial relative co-Hopf property for hyperbolic groups.

Proposition 4.24: *Let G be a torsion-free hyperbolic group. Let H be a non-cyclic subgroup of G, with respect to which G is freely indecomposable. There exists a finite subset F_0 of H such that if $\phi : G \to G$ is an injective morphism which fixes F_0, then it is an isomorphism.*

Proof. Just take F_0 to be a generating set for the subgroup H_0 given by Lemma 4.23. If ϕ fixes F_0, it fixes H_0 relative to which G is freely indecomposable. Thus we can apply Proposition 4.22 to G with the subgroup H_0, to deduce that ϕ is an isomorphism. □

5 Elementary embeddings in hyperbolic groups

5.1 Hyperbolic towers

We define hyperbolic towers.

Definition 5.1: (hyperbolic floor) *Consider a triple (G, G', r) where G is a group, G' is a subgroup of G, and r is a retraction from G onto G'. We say that (G, G', r) is a hyperbolic floor if there exists a non-trivial decomposition Λ of G as a graph of groups with surfaces (recall Definition 4.1) such that:*

- *G' is the subgroup of G generated by the non surface type vertex groups of Λ, and it is in fact their free product;*
- *every edge of Λ joins a surface type vertex to a non surface type vertex (bipartism);*
- *the retraction r sends surface type vertex group of Λ to non-abelian images.*

Definition 5.2: (hyperbolic tower) *Let G be a group, let H be a subgroup of G. We say that G is a hyperbolic tower based on H if there exists a finite sequence $G = G^0 \geq G^1 \geq \ldots \geq G^m \geq H$ of subgroups of G where $m \geq 0$ and:*

- *for each k in $[0, m-1]$, there exists a retraction $r_k : G^k \to G^{k+1}$ such that the triple (G^k, G^{k+1}, r_k) is a hyperbolic floor, and H is contained in one of the non surface type vertex group of the corresponding hyperbolic floor decomposition;*
- *$G^m = H * F * S_1 * \ldots * S_p$ where F is a (possibly trivial) free group, $p \geq 0$, and each S_i is the fundamental group of a closed surface without boundary of Euler characteristic at most -2.*

Remark 5.3: *If G_1 and G_2 are hyperbolic towers over subgroups H_1 and H_2, then $G_1 * G_2$ is a hyperbolic tower over $H_1 * H_2$. If G is a hyperbolic tower over a subgroup G', and G' is a hyperbolic tower over a subgroup H, then G is a hyperbolic tower over H.*

Recall that our main result, Theorem 1.2, says that if G is a torsion-free hyperbolic group, and H is an elementary subgroup of G, then G is a hyperbolic tower based on H.

To prove Theorem 1.2, we need to construct successive retractions from subgroups of G to proper subgroups until we get to H. The strategy will be to build, by the mean of first-order sentences, some maps that we will call preretractions, and which preserve some characteristics of the cyclic JSJ decomposition of these subgroups of G, or of their relative cyclic JSJ decomposition with respect to H. Then we will show that the existence of a preretraction implies the existence of a hyperbolic floor.

5.2 JSJ decompositions

A JSJ decomposition Λ of a group G is a decomposition as a graph of groups which encodes all possible splittings of the group G over a given class $\mathcal{E}$ of subgroups. The standard reference for the case where G is finitely presented and one-ended, and $\mathcal{E}$ is the class of finite and cyclic groups is [RS97]. This has been generalised in [DS99] and [FP06] to the case where $\mathcal{E}$ is the class of slender groups. In the case where G is one-ended hyperbolic, [Bow98] gives a canonical construction. A JSJ decomposition of a group G relative to a subgroup H is a graph of groups decomposition in which H is elliptic, and which encodes all possible splittings of G in which H is elliptic and edge groups lie in $\mathcal{E}$.

In the sequel, we will use the JSJ decomposition in the case where G is torsion-free hyperbolic and freely indecomposable (respectively freely indecomposable with respect to a subgroup H), and $\mathcal{E}$ is the class of cyclic groups. We call such a decomposition the cyclic JSJ decomposition of G (respectively the relative cyclic JSJ decomposition with respect to H). In this case, the (relative) cyclic JSJ decomposition admits a natural structure of graph of groups with surfaces. Moreover, in the relative case, the subgroup H lies in a non surface type vertex group.

We will need only a few properties of such a cyclic (relative) JSJ decomposition Λ: the most important is that its vertex groups are 'preserved' under modular automorphisms, as given by

Lemma 5.4: *Let G be a torsion-free hyperbolic group which is freely indecomposable. Denote by Λ its cyclic JSJ decomposition, as given by Theorem 7.1 of [RS97]. An element of $\mathrm{Mod}(G)$ restricts to conjugation on each non surface type vertex group of Λ, and sends surface type vertex groups isomorphically on conjugates of themselves.*

Similarly, suppose G is a torsion-free hyperbolic group which is freely indecomposable with respect to a subgroup H. Let Λ denote its cyclic relative JSJ decomposition with respect to H. An element of $\mathrm{Mod}_H(G)$ restricts to conjugation on each non surface type vertex group of Λ, and sends surface type vertex groups isomorphically on conjugates of themselves.

This lemma is a consequence of the universal property of the cyclic (relative) JSJ decomposition: recall that the modular group $\mathrm{Mod}(G)$ is generated by automorphisms of G which preserve some cyclic splitting of G, and that the JSJ decomposition in some sense contains all such splitting.

The other properties of a cyclic (relative) JSJ decomposition Λ of a torsion-free hyperbolic group we will use are summarized in the following remark. Call Z type vertices the vertices of Λ which have infinite cyclic vertex group, and call rigid type vertices the vertices which are neither of Z type, nor of

surface type. We will also say that a vertex in the tree T_Λ is of type Z or rigid according to the type of its image by the quotient map $T_\Lambda \to \Lambda$.

Remark 5.5: *Let Λ be a cyclic JSJ decomposition (relative to a subgroup H) of a torsion-free hyperbolic group G which is freely indecomposable (relative to the subgroup H). Then*

(i) the edge groups of Λ are cyclic;

(ii) an edge of Λ is adjacent to at most one surface type vertex, and to at most one Z type vertex;

(iii) (strong 2-acylindricity) if a non-trivial element of A stabilises two distinct edges of T_Λ, they are adjacent and their common endpoint is a Z type vertex.

Not all the decompositions given by Theorem 7.1 of [RS97] satisfy these properties, but we can easily find one that does: see for example the decomposition given by Theorem 5.28 of [Bow98].

A lot of the results we will need about cyclic (relative) JSJ decompositions only use the properties given by Remark 5.5. We thus give

Definition 5.6: (JSJ-like decomposition) *Let Λ be a graph of groups with surfaces, with fundamental group A. Call Z type vertices the vertices of Λ which have infinite cyclic groups, and rigid type vertices the vertices which are neither Z type, nor surface type. We say that Λ is a JSJ-like decomposition of A if it satisfies the properties (i), (ii) and (iii) given in Remark 5.5.*

Thus in particular a (relative) cyclic JSJ decomposition is a JSJ-like decomposition. Note that distinct vertices of the tree corresponding to a JSJ-like decomposition have distinct stabilisers.

Definition 5.7: (subgroups with disjoint conjugacy classes) *We say that two subgroups of A have disjoint conjugacy classes if no non-trivial element of one of the subgroups has a conjugate in the other.*

Remark 5.8: *Given a strongly 2-acylindrical graph of groups decomposition Λ of a group A,*

(i) if G_1 and G_2 are stabilisers of edges e_1 and e_2 of T_Λ, they have disjoint conjugacy classes in A unless either $e_1 = e_2$, or e_1 and e_2 are adjacent to a common Z type vertex;

(ii) if the edge group corresponding to an edge e of T_Λ is not maximal cyclic in A, then e is adjacent to a Z type vertex.

5.3 Preretractions

Preretractions are morphisms that preserve some of the structure of a JSJ-like decomposition. We need to define them as maps $A \to G$ where A is a subgroup of G.

Definition 5.9: (preretraction) *Let G be a group, let A be a subgroup of G, and let Λ be a JSJ-like decomposition of A. A morphism $A \to G$ is a preretraction with respect to Λ if its restriction to each non surface type vertex group A_v of Λ is just a conjugation by some element g_v of G, and if surface type vertex groups have non-abelian images.*

Remark 5.10: *In a JSJ-like decomposition, every edge group is contained in a non surface type vertex group, thus the restriction of a preretraction to an edge group is just a conjugation by an element of G.*

We will now give two results which are central in our proof of Theorem 1.2.

Proposition 5.11: *Let A be a torsion-free hyperbolic group. Let Λ be a cyclic JSJ-like decomposition of A. Assume that there exists a non-injective preretraction $A \to A$ with respect to Λ. Then there exists a subgroup A' of A, and a retraction r from A to A', such that (A, A', r) is a hyperbolic floor. Moreover, given a rigid type vertex group R_0 of Λ, we can choose A' to contain R_0.*

The second proposition says in which case we can get a preretraction $A \to A$ from a preretraction $A \to G$, so that we can then apply Proposition 5.11. It will be needed for the induction steps in the proof of 1.2.

Proposition 5.12: *Let G be a torsion-free hyperbolic group. Let A be a non-cyclic retract of G which admits a cyclic JSJ-like decomposition Λ. Suppose G' is a subgroup of G containing A such that either G' is a free factor of G, or G' is a retract of G by a retraction $r : G \to G'$ which makes (G, G', r) a hyperbolic floor. If there exists a non-injective preretraction $A \to G$ with respect to Λ, then there exists a non-injective preretraction $A \to G'$ with respect to Λ.*

The last three sections of this paper are devoted to the proofs of these two propositions: they are both intermediate steps in the proof of Proposition 6 of [Sel06] but are not explicitly stated there. For now, we will assume these two results hold, and use them to prove Theorem 1.2.

5.4 Using first order to build preretractions

Suppose H is a group with an elementary embedding in a torsion-free hyperbolic group G. To show that G admits a structure of hyperbolic tower over H, we will start by decomposing G in free factors relatively to H. That is, we will write $G = A * B_1 \ldots * B_m$ where H is contained in A, the groups B_j are freely indecomposable (possibly infinite cyclic) and A is freely indecomposable with respect to H. We call such a decomposition a Grushko decomposition of G relative to H.

If we can show that A admits a structure of hyperbolic tower over H, and that the groups B_i admit structures of hyperbolic towers over 1, we will be done by Remark 5.3. The idea is thus to produce non-injective preretractions $A \to A$ and $B_i \to B_i$, in order to be able to apply Proposition 5.11 and get the top floor of a hyperbolic tower decomposition. But for this, it is enough by Proposition 5.12 to build non-injective preretractions $A \to G$ and $B_i \to G$.

This is what the two following results will enable us to do. In fact they are slightly more general: this greater generality is required for the induction step, when we will build further floors of our hyperbolic towers.

Proposition 5.13: *Suppose that G is a non-cyclic torsion-free hyperbolic group, and let H be a subgroup whose embedding in G is elementary. Suppose A is a subgroup of G which is hyperbolic, properly contains H, and is freely indecomposable relative to H. Let Λ be the cyclic JSJ decomposition of A relative to H. Then there exists a non-injective preretraction $A \to G$ with respect to Λ.*

Proposition 5.14: *Suppose that G is a torsion-free hyperbolic group, and that H is a subgroup elementarily embedded in G which is also a retract of G. Let B be a freely indecomposable hyperbolic subgroup of G which is neither cyclic nor a closed surface group of Euler characteristic at most -2. Let Λ be the cyclic JSJ decomposition of B. Suppose that no non-trivial element of B is conjugate in G to an element of H. Then there exists a non-injective preretraction $B \to G$ with respect to Λ.*

The heart of the proof of Theorem 1.2 is contained in these two results. The idea of the proof of Propositions 5.13 and 5.14 is to express (a consequence of) the existence of a factor set by a first-order logic formula satisfied by H whose interpretation on G then gives us a non-injective preretraction.

We will use the following definition

Definition 5.15: (Λ-related morphisms) *Let A be a group which admits a JSJ-like decomposition Λ, and let f be a morphism from A to a group G. We say that a morphism $f' : A \to G$ is Λ-related to f if*

- *for each non surface type vertex group R of Λ, there exists an element u_R such that f' restricted to R is $\mathrm{Conj}(u_R) \circ f$;*
- *each surface type vertex group of Λ which has non-abelian image by f also has non-abelian image by f'.*

Remark 5.16: *Suppose A is a subgroup of a group G, that it admits a JSJ-like decomposition Λ, and let $f : A \to G$ be a morphism. Then*

- *f is Λ-related to the embedding $A \hookrightarrow G$ if and only if it is a preretraction with respect to Λ;*
- *if $\sigma \in \mathrm{Mod}(G)$, then $f' = f \circ \sigma$ is Λ-related to f by Lemma 5.4.*

The following lemma shows that Λ–relatedness can be expressed in first-order logic.

Lemma 5.17: *Let A be a group generated by a finite tuple $\mathbf{a}$. Suppose A admits a JSJ-like decomposition Λ. There exists a formula $\mathrm{Rel}(\mathbf{x}, \mathbf{y})$ such that for any pair of morphisms f and f' from A to G, the formula $\mathrm{Rel}(f(\mathbf{a}), f'(\mathbf{a}))$ is satisfied by G if and only if f' is Λ-related to f.*

Proof. We introduce some notation. Denote by $R_1, \ldots R_r$ the non surface type vertex groups of Λ, and by $S_1, \ldots S_s$ its surface type vertex groups. For $1 \leq p \leq r$, choose a finite generating set $\boldsymbol{\rho}_p$ for R_p, and for $1 \leq q \leq s$, choose a finite generating set $\boldsymbol{\sigma}_q$ for S_q. We take the convention to denote tuples by bold font, and to denote by $l(\mathbf{x})$ the cardinality of the tuple $\mathbf{x}$.

The elements of $\boldsymbol{\rho}_p$ and $\boldsymbol{\sigma}_q$ can be represented by words in the elements $\mathbf{a}$, we denote these by $\boldsymbol{\rho}_p = \bar{\boldsymbol{\rho}}_p(\mathbf{a})$ and $\boldsymbol{\sigma}_q = \bar{\boldsymbol{\sigma}}_q(\mathbf{a})$ respectively.

Now, if w is an element of A which can be represented by a word $\bar{w}(\mathbf{a})$, its image by the morphism $f : A \to G$ extending $\mathbf{a} \mapsto \mathbf{g}$ is represented by $\bar{w}(\mathbf{g})$. Thus we have

$$\begin{aligned} f(\boldsymbol{\rho}_p) &= \bar{\boldsymbol{\rho}}_p(\mathbf{g}) \\ f(\boldsymbol{\sigma}_q) &= \bar{\boldsymbol{\sigma}}_q(\mathbf{g}) \end{aligned}$$

The maps f and f' extending $\mathbf{a} \mapsto \mathbf{g}$ and $\mathbf{a} \mapsto \mathbf{g}'$ respectively satisfy the relatedness condition on the rigid type vertex groups of Λ if and only if

$$\exists u_1 \ldots \exists u_r \bigwedge_{p=1}^{r} \left\{ \bar{\boldsymbol{\rho}}_p(\mathbf{g}') = u_p \bar{\boldsymbol{\rho}}_p(\mathbf{g}) u_p^{-1} \right\}.$$

To express the abelianity of a subgroup generated by a tuple $\mathbf{z} = (z^1, \ldots, z^{l(\mathbf{z})})$, we can use the formula $\mathrm{Ab}(\mathbf{z}) : \bigwedge_{i,j} \left\{ [z^i, z^j] = 1 \right\}$. Thus the non-abelianity condition about the image by f and f' of surface type vertex groups of Λ can also be expressed by

$$\bigwedge_{q=1}^{s} \left\{ \neg \mathrm{Ab}(\bar{\boldsymbol{\sigma}}_q(\mathbf{g})) \Rightarrow \neg \mathrm{Ab}(\bar{\boldsymbol{\sigma}}_q(\mathbf{g}')) \right\}.$$

Consider the formula $\mathrm{Rel}(\mathbf{x}, \mathbf{y})$ with free variables $\mathbf{x}, \mathbf{y}$ given by

$$\exists u_1 \ldots \exists u_r \left[\bigwedge_{p=1}^{r} \left\{ \bar{\boldsymbol{\rho}}_p(\mathbf{y}) = u_p \bar{\boldsymbol{\rho}}_p(\mathbf{x}) u_p^{-1} \right\} \right] \wedge \left[\bigwedge_{q=1}^{s} \left\{ \neg \mathrm{Ab}(\bar{\boldsymbol{\sigma}}_q(\mathbf{x})) \Rightarrow \neg \mathrm{Ab}(\bar{\boldsymbol{\sigma}}_q(\mathbf{y})) \right\} \right].$$

Then if $f : \mathbf{a} \mapsto \mathbf{g}$ and $f' : \mathbf{a} \mapsto \mathbf{g}'$, the sentence $\mathrm{Rel}(\mathbf{g}, \mathbf{g}')$ is satisfied by G if and only if f' is Λ-related to f. $\square$

We can now prove the two key propositions.

Proof of Proposition 5.13. By Proposition 4.16, there exists a finite subset H_0 of H, and a finite family of proper quotients $\eta_j : A \to L_j$ for $j \in [1, m]$, such that any non-injective morphism $\theta : A \to A$ which fixes H_0 factors through one of the quotients η_j after precomposition by an element of $\mathrm{Mod}_H(A)$. Proposition 4.24 on the other hand shows that an injective morphism $\theta : A \to A$ which fixes a big enough finite subset of H is also surjective.

Now a morphism $\theta : A \to H$ can be seen as a non-surjective morphism $A \to A$ since we assumed $H \neq A$. Thus, up to enlarging H_0, any morphism $\theta : A \to H$ which fixes H_0 is non-injective, so there exists an element τ of $\mathrm{Mod}_H(A)$ such that $\theta' = \theta \circ \tau$ factors through one of the quotients η_j.

Let Λ be the cyclic JSJ decomposition of A relative to H. By Remark 5.16, we can weaken the previous statement to saying that for any morphism $\theta : A \to H$ which fixes H_0, there exists a morphism $\theta' : A \to H$ which is Λ-related to θ and which factors through one of the quotients η_j.

Finally, for each l in $[1, m]$, we fix an element ν_j in the kernel of $\eta_j : A \to Q_l$. We further weaken the above statement to get

Statement 1: *For any morphism $\theta : A \to H$ which fixes H_0, there exists a morphism $\theta' : A \to H$ which is Λ-related to θ, and an index j in $[1, m]$ such that $\theta'(\nu_j) = 1$.*

We claim that these successive weakenings have ensured that Statement 1 can be expressed by a first order sentence in the language $\mathcal{L}_H$ which is satisfied by H.

The group A is hyperbolic, we choose a finite presentation $\langle \mathbf{a} \mid \bar{\mathbf{\Sigma}}_A(\mathbf{a})\rangle$. If an $l(\mathbf{a})$-tuple $\mathbf{x}$ in H satisfies $\bar{\mathbf{\Sigma}}_A(\mathbf{x}) = 1$, the map $A \to H$ which sends $\mathbf{a}$ to $\mathbf{x}$ is a morphism. Conversely, any morphism $A \to H$ comes from a solution to the system of equations $\bar{\mathbf{\Sigma}}_A(\mathbf{x}) = 1$ in H. The elements ν_l can be represented by words $\bar{\nu}_l(\mathbf{a})$; and for each h in H_0, the element h can be represented by a word $\bar{h}(\mathbf{a})$.

Recall that the language $\mathcal{L}_H$ is defined as the language of groups to which we have added a constant symbol $\lceil h \rceil$ for each h in H. To express that the morphism $A \to H$ extending $\mathbf{a} \mapsto \mathbf{x}$ fixes the finite subset H_0 of H, we can thus write

$$\bigwedge_{h \in H_0} \left\{ \lceil h \rceil = \bar{h}(\mathbf{x}) \right\}.$$

To express that the morphism corresponding to the tuple $\mathbf{x}$ sends one of the elements ν_i to 1, we can write

$$\bigvee_{i=1}^{m} \left\{ \bar{\nu}_i(\mathbf{x}) = 1 \right\}.$$

Finally consider the sentence (†) over $\mathcal{L}_H$ given by

$$\forall \mathbf{x} \left[\bar{\Sigma}_A(\mathbf{x}) = 1 \wedge \bigwedge_{h \in H_0} \lceil h \rceil = \bar{h}(\mathbf{x}) \right] \Rightarrow \exists \mathbf{y} \left[\bar{\mathbf{\Sigma}}_A(\mathbf{y}) = 1 \right] \wedge \mathrm{Rel}(\mathbf{x}, \mathbf{y}) \wedge \left[\bigvee_{l=1}^{m} \bar{\nu}_l(\mathbf{y}) = 1 \right].$$

The interpretation of the first-order formula (†) on H is exactly Statement 1, so it is true on H. The formula (†) is therefore satisfied by G. Let us look at its interpretation on G.

If we take the 'tautological solution' $\mathbf{a}$ to the equation $\bar{\Sigma}_A(\mathbf{x}) = 1$, it satisfies the formula in the first square brackets: indeed, $\bar{\Sigma}_A(\mathbf{a}) = 1$, and for each $h \in H_0$, we have $h = \bar{h}(\mathbf{a})$ by definition of $\bar{h}$. Thus, by the second part of the formula, we get a tuple $\mathbf{y}$ such that $\mathbf{a} \mapsto \mathbf{y}$ extends to a morphism $\mu : A \to G$, which is Λ-related to the morphism $\mathbf{a} \mapsto \mathbf{a}$, and which sends one of the elements ν_i to 1. In particular, it is not injective. But the morphism $\mathbf{a} \mapsto \mathbf{a}$ is just the embedding $A \hookrightarrow G$, so by Remark 5.16, $\mu : A \to G$ is a non-injective preretraction. □

We now show the second key result.

Proof of 5.14. Assume first that B is not the fundamental group of the closed surface of Euler characteristic -1. We choose a presentation $\langle \mathbf{b} \mid \bar{\mathbf{\Sigma}}_B(\mathbf{b})\rangle$ for B. Let Λ be the cyclic JSJ decomposition of B.

Since H is a retract of G, it is a quasiconvex, thus it is itself hyperbolic (see Proposition 4.2, Chapter 10 of [CDP90]). By Proposition 4.6, there exist proper quotients $\eta_1 : B \to L_1, \ldots, \eta_m : B \to L_m$ of B such that any non-injective morphism $B \to H$ factors through one of the maps η_j after precomposition by an element of $\mathrm{Mod}(B)$. Again we choose non-trivial elements $\nu_1, \ldots \nu_m$ of B such that ν_j is in the kernel of η_j.

If we proceed to the same weakenings as in the proof of Proposition 5.13, we see that for any non-injective morphism θ from B to H, there is another morphism θ' which is Λ-related to θ, and which kills one of the ν_j.

We now need to find a sufficient condition for non-injectivity of a map $B \to H$ that is expressible in first-order. Proposition 4.14, applied to B and H, tells us that there exist a finite set $i_1, \ldots i_t$ of embeddings of B in H such that for any embedding $i : B \hookrightarrow H$, there exists an element σ of $\mathrm{Mod}(B)$, an integer k in $[1, t]$ and an element h of H such that

$$\mathrm{Conj}\,(h) \circ i \circ \sigma = i_k$$

By Remark 5.16, the map on the left hand side is Λ-related to i.

Let $\theta : B \to H$. Consider the following statement

S(θ): *Suppose that $\theta'' : B \to H$ is a morphism which is Λ-related to θ. Then for any integer k in $[1,t]$, we have $\theta''(\mathbf{b}) \neq i_k(\mathbf{b})$.*

From the previous paragraph, if $S(\theta)$ holds, then θ isn't an embedding: it is a sufficient condition for a map not to be an embedding. We finally get that the following statement holds.

Statement 2: *If $\theta : B \to H$ is a morphism for which $S(\theta)$ holds, then there exists a morphism $\theta' : B \to H$ which is Λ-related to θ, and an integer l in $[1,m]$ such that $\theta'(\nu_l) = 1$.*

We claim that this can be expressed by a first-order formula. Let us first express $S(\theta)$ by a first order formula on the variables $\theta(\mathbf{b})$. Consider the following first order formula $\psi(\mathbf{x})$ with free variable the $l(\mathbf{b})$-tuple $\mathbf{x}$:

$$\left[\bar{\mathbf{\Sigma}}_B(\mathbf{x}) = 1\right] \wedge \forall \mathbf{z} \left[\bar{\mathbf{\Sigma}}_B(\mathbf{z}) = 1 \wedge \mathrm{Rel}(\mathbf{z},\mathbf{x})\right] \Rightarrow \left[\bigwedge_{k=1}^{t} \mathbf{z} \neq \lceil i_k(\mathbf{b}) \rceil\right].$$

This is a first order formula in the language $\mathcal{L}_H$. The constant $\lceil i_k(\mathbf{b})\rceil$ is interpreted in both H and G simply by the element $i_k(\mathbf{b})$.

Let $\mathbf{x}$ be a $l(\mathbf{b})$-tuple in H. It is straightforward to see that the formula $\psi(\mathbf{x})$ is satisfied by H if and only if the map $\mathbf{b} \mapsto \mathbf{x}$ extends to a morphism θ for which the statement $S(\theta)$ holds. So if $\psi(\mathbf{x})$ is satisfied by H, the map $\mathbf{b} \mapsto \mathbf{x}$ extends to a non-injective morphism $\theta : B \to H$.

We can now write the first order sentence (††)

$$\forall \mathbf{x}\, \psi(\mathbf{x}) \;\Rightarrow\; \exists \mathbf{y} \left[\bar{\mathbf{\Sigma}}_B(y) = 1\right] \wedge \mathrm{Rel}(\mathbf{x},\mathbf{y}) \;\wedge\; \left[\bigvee_{j=1}^{l} \bar{\nu}_j(\mathbf{y}) = 1\right].$$

The first order formula (††) on H expresses Statement 2, so it is satisfied by H.

As H is elementarily embedded in G, the formula (††) is also satisfied by G. As in the proof of 5.13, we can apply it to the tautological solution $\mathbf{b}$ of $\bar{\mathbf{\Sigma}}_B(\mathbf{x}) = 1$. To see that $G \models \psi(\mathbf{b})$, note first that since we assumed that B is not the fundamental group of a closed surface, the JSJ decomposition of B admits at least one non-surface type vertex group R. A map $\mu : B \to G$ which is Λ-related to the embedding $\mathbf{b} \mapsto \mathbf{b}$ restricts to conjugation on R, thus $\mu(R)$ cannot lie in H since by hypothesis, no element of B can be conjugated into H by an element of G. This implies in particular that for all k, the $l(\mathbf{b})$-tuple $\mu(\mathbf{b})$ is distinct from the tuple $i_k(\mathbf{b})$, so that $G \models \psi(\mathbf{b})$.

The second part of the sentence (††) thus gives a morphism $B \to G$ which is Λ-related to the embedding $B \hookrightarrow G$ and kills one of the elements ν_i: it is a non-injective preretraction.

In the case where B is the fundamental group of the non-orientable surface of Euler characteristic -1, we can follow the same proof if we consider the JSJ of B to consist of a single non surface type vertex, and $\mathrm{Mod}(B)$ to be trivial. Indeed, the group of automorphisms of B is finite, so if we replace the finite list $(\eta_i)_i$ by the finite list $(\eta_i \circ \tau)_{i,\tau \in \mathrm{Aut}(B)}$, the conclusions of 4.6 still hold with our new definition of the modular group. Similarly, the existence of a finite set of embeddings up to conjugation holds. □

5.5 Proof of the main result

Assuming we have Proposition 5.11 and Proposition 5.12, we can now prove Theorem 1.2.

Proof. Note first that if G is infinite cyclic, its only elementarily embedded subgroup is itself, and the theorem is trivial.

Assume thus that G is a non-cyclic torsion-free hyperbolic group, and let H be an elementary subgroup of G. Note that H is necessarily non-abelian as it is elementary equivalent to G.

We will first show that G admits a structure of hyperbolic tower over a group G' whose Grushko decomposition relative to H is of the form $G' = H * B'_1 * \ldots * B'_r$. Once this is done, we will show that each of the B'_i has a structure of hyperbolic tower over $\{1\}$, and this will give the result by Remark 5.3.

Set $G^0 = G$. We define by induction a finite sequence $G = G^0 > G^1 > \ldots > G^N$ of subgroups of G, such that H is a free factor of G^N, and G^m has a structure of hyperbolic floor over G^{m+1} for each m up to $N-1$.

Assume G^m is defined, and write the Grushko decomposition of G^m relative to H as

$$G^m = A^m * B_1^m * \ldots * B_{p_m}^m$$

where A^m is the factor containing H. If $A^m = H$ we are done, so assume $A^m \neq H$.

Note that A^m is freely indecomposable relative to H, and that, as a retract, it is a quasiconvex subgroup of G, and thus it is hyperbolic. Denote by Λ the cyclic JSJ decomposition of A^m relative to H.

All the hypotheses of Proposition 5.13 for $A = A^m$ are satisfied, so we can apply it to get a non-injective preretraction $A^m \to G$ with respect to Λ. We now apply Proposition 5.12 successively to the floors of the hyperbolic tower formed by G over G^m, and to the free product $G^m = A^m * (B_1^m * \ldots * B_{p_m}^m)$, to get a non-injective preretraction $A^m \to A^m$ with respect to Λ. Finally by Proposition 5.11, we get a retraction $r : A^m \to A_0^m$ on a proper subgroup of A^m such that (A^m, A_0^m, r) is a floor of a hyperbolic tower, and the rigid group of Λ which contains H is in A_0^m. Now define G^{m+1} by

$$G^{m+1} = A_0^m * B_1^m * \ldots * B_{p_m}^m.$$

Since A^m has a structure of hyperbolic floor over A_0^m, the group G^m has a structure of hyperbolic floor over G^{m+1} as required.

For each m, the group G^{m+1} is a strict retract of G^m, and since the groups G^m are all subgroups of G, they are G-limit groups. Thus by Proposition 4.10 the sequence is finite. At the end of this process, we get a group G^N in which H is a free factor, and such that G is built as a hyperbolic tower based on G^N.

If all the other factors of the Grushko decomposition of G^N relative to H are surface groups or free groups, we are done. So assume that there is a factor B which is neither free nor a closed surface group. Note that as a retract of G, the group G^N is hyperbolic, so as a free factor of G^N, the group B is itself hyperbolic. We will now show that B has a structure of hyperbolic tower over $\{1\}$.

Both H and B are free factors of G^N, thus any two of their conjugates in G^N intersect trivially. But since G^N is a retract of G, any two conjugates of H and B in G must also intersect trivially. Hence the conditions of 5.14 are satisfied by B: by applying it, we get a non-injective preretraction $B \to G$. We apply 5.12 iteratively to get a non-injective preretraction $B \to B$, which by 5.11 gives us a retraction $r : B \to B'$, such that (B, B', r) is a hyperbolic floor.

Note that since B' is a retract of G, the number of factors in its Grushko decomposition is bounded above by the rank of G. If any of the factors of the Grushko decomposition of B' are neither free nor surface, we can repeat the process above. This terminates, as before, because the groups involved are G-limit groups and because the number of factors in the Grushko decomposition of our groups is bounded. We finally get that B is a hyperbolic tower over $\{1\}$.

Thus all the factors of G^N distinct from H are hyperbolic towers over $\{1\}$. By Remark 5.3, the group G^N is a hyperbolic tower over H. This terminates the proof. □

5.6 The case of free groups

In the special case where our hyperbolic group is free, we can use Theorem 1.2 to show Theorem 1.3. To do so, we need only prove that an elementary subgroup of a free group is a non-abelian free factor, since the converse is given by Theorem 4 in [Sel06].

Proof of Theorem 1.3. Suppose that H is an elementary subgroup of F. By Theorem 1.2, F has a structure of hyperbolic tower over H. If the tower has at least one floor, there exists a subgroup F' of F, and a retraction $r : F \to F'$ so that $H \leq F'$, and (F, F', r) is a hyperbolic floor built by amalgamating some surface groups with boundary to free factors of F. Pick generators $\gamma_1, \ldots \gamma_r$ of a maximal set of pairwise non-conjugate maximal boundary subgroups of these surface groups. We know,

from the standard presentation of a surface group with boundary, that the product of the elements γ_i is equal in F to a product of commutators and squares.

By Lemma 4.1 in [BF94], since both F and F' are free groups, there is a decomposition of F' as $Z * F''$ where Z is an infinite cyclic group generated by one of these maximal boundary elements (say γ_1), and all the other maximal boundary elements γ_i lie in conjugates of F''. Now let $\alpha : F' \to Z/2Z$ be the map which kills F'' and the squares in Z. The image of γ_1 by $\alpha \circ r$ is the generator of $Z/2Z$, and for $i \neq 1$, the image of γ_i is trivial. However, squares and commutators of F are sent to 1 by $\alpha \circ r$, so we get a contradiction. This shows that the only structure of hyperbolic tower that a free group can have over one of its subgroup is a trivial one, where the subgroup is a free factor of the free group. Thus H is a free factor of F. □

5.7 The case of surface groups

All the surfaces we consider are compact and connected. We show

Theorem 5.18: *Let S be the fundamental group of a surface without boundary Σ such that $\chi(\Sigma) \leq -1$. If H is a proper elementary subgroup of S, then it is a free factor of the fundamental group of a minor subsurface of Σ.*

Minor subsurfaces are defined by

Definition 5.19: *Let Σ be a closed surface. Let Σ_0 be a proper connected subsurface of Σ. Denote by Σ_1 the closure of $\Sigma \setminus \Sigma_0$. We say that Σ_0 is a minor subsurface of Σ if*

- *Σ_1 is connected;*
- *$\chi(\Sigma_0) \geq \chi(\Sigma)/2$, with equality if and only if Σ_0 and Σ_1 are homeomorphic;*
- *if Σ_0 is non-orientable, so is Σ_1.*

Note that Theorem 1.4 that we gave in the introduction is a consequence of Theorem 5.18.

To prove Theorem 5.18, we apply Theorem 1.2, which tells us that S has a structure of hyperbolic tower over H, and then we use the "only if" part of the following equivalence result.

Proposition 5.20: *Let S be the fundamental group of a surface Σ without boundary. Then S admits a structure of hyperbolic tower over a proper subgroup H if and only if H is a free factor of the fundamental group of a minor subsurface of Σ.*

Clearly, it is enough to prove Proposition 5.20. Suppose S has a structure of hyperbolic tower over a subgroup H. Note that S is freely indecomposable, so if $H \neq S$, the tower contains at least one hyperbolic floor. Thus there exists a retraction r from S to a subgroup R containing H such that (S, R, r) is a hyperbolic floor, and R itself has a structure of hyperbolic tower over H. Denote by Γ the graph of groups decomposition associated to the hyperbolic floor structure (S, R, r). Denote its non surface type vertex groups by $R_0, \ldots, R_k$, and its surface type vertex groups by $S_1, \ldots, S_l$. Recall that R is the free product of the subgroups R_j, and that we may assume $H \leq R_0$.

By Theorem III.2.6 of [MS84], the splitting Γ of S is geometric, namely it is dual to a set of non null-homotopic simple closed curves on the surface Σ. Thus each of the groups S_i, R_j is the fundamental group of a connected subsurface of Σ that we denote by Σ_i, Ξ_j respectively. Denote by r_i the restriction of r to S_i for $1 \leq i \leq l$.

Remark 5.21: *Suppose that for some i, the map r_i is non-pinching with respect to Σ_i (recall Definition 3.9). Then $r_i(S_i)$ lies in a conjugate of one of the subgroups R_j in R.*

Indeed, consider the R-tree T_* with trivial edge stabilisers corresponding to the free factor decomposition $R = R_0 * \ldots * R_k$. The group S_i acts on this tree via r_i, and in this action, its boundary elements are elliptic: by Lemma 3.5, we get a set of simple closed curves $\mathcal{C}$ on Σ_i whose corresponding element stabilise edges of T_*, and thus have trivial image by r_i. Since r_i is non-pinching, $\mathcal{C}$ is empty and $r_i(S_i)$ is elliptic in T_*. Thus, it is contained in a conjugate in R of one of the groups R_j. This proves the claim.

Thus if for some i, the map r_i is non-pinching with respect to Σ_i, it is a morphism of surface groups between S_i and R_j. Moreover, it sends non-conjugate maximal boundary elements of S_i to non-conjugate maximal boundary elements of R_j. Here is a result about such maps. It is a consequence of Theorem 3.8 and Lemma 3.10.

Lemma 5.22: *Suppose that $f : S_1 \to S_0$ is a morphism between fundamental groups of surfaces with boundaries Σ_1 and Σ_0 which sends non-conjugate maximal boundary elements to non-conjugate maximal boundary elements and is non-pinching. Then f is an isomorphism of surface groups.*

Proof. There exists a continuous map $\phi : \Sigma_1 \to \Sigma_0$ such that $\phi_* = f$. The properties of f imply that we can choose ϕ to be injective on $\partial\Sigma_1$, to satisfy $\phi(\partial\Sigma_1) \subseteq \partial\Sigma_0$, and to send non null-homotopic simple closed curve on Σ_1 to non null-homotopic images.

Suppose f is not injective: by Theorem 3.8, there is some non boundary-parallel simple arc α in Σ_1 such that $\phi(\alpha)$ is boundary parallel. Since ϕ is injective on $\partial\Sigma_1$, the endpoints of α must belong to the same boundary component of Σ_1, so they are joined by two non-homotopic paths β and β' which lie in this boundary component. Since ϕ sends $\partial\Sigma_1$ into $\partial\Sigma_0$, we see that the image of either $\alpha\beta$ or $\alpha\beta'$ by ϕ is null-homotopic in Σ_0: this contradicts the properties of ϕ. We deduce that f is injective.

Now f is non-pinching, so by Lemma 3.10, the subgroup $f(S_1)$ has finite index in S_0. Thus there is a finite cover $p : \tilde{\Sigma}_0 \to \Sigma_0$ of Σ_0 such that $p_*(\tilde{S}_0) = f(S_1)$, where $\tilde{S}_0 = \pi_1(\tilde{\Sigma}_0)$. The map $p_*^{-1} \circ f$ is an isomorphism between S_1 and $\tilde{S}_0$ seen as free groups (without their surface group structure). Moreover, it sends maximal boundary elements of S_1 to maximal boundary elements of $\tilde{S}_0$.

Let M contain a maximal boundary element corresponding to each boundary component of Σ_1: it does not extend to a basis of S_1 as a free group. Thus the image of M by $p_*^{-1} \circ f$ is a set of maximal boundary elements of $\tilde{S}_0$ which does not extend to a basis of $\tilde{S}_0$. This is only possible if $p_*^{-1} \circ f(M)$ contains an element corresponding to each boundary component of $\tilde{\Sigma}_0$, so we get $|\partial\tilde{\Sigma}_0| \leq |\partial\Sigma_1|$.

Now f is injective on the boundary of Σ_1 so that $|\partial\Sigma_1| \leq |\partial\Sigma_0|$; and $\tilde{\Sigma}_0$ is a finite cover of Σ_0, so $|\partial\Sigma_0| \leq |\partial\tilde{\Sigma}_0|$, with equality if and only if p is a homeomorphism.

All these inequalities must therefore be equalities, so p_* is an isomorphism and $f(S_1) = p_*(\tilde{S}_0) = S_0$, and Σ_0 has the same number of boundary components as Σ_1. This implies first that f is surjective, then that it is in fact an isomorphism of surface groups. □

Thus if we are in the hypotheses of Proposition 5.20, and if at least one of the maps $r_i = r|_{S_i}$ is non-pinching, Remark 5.21 and Lemma 5.22 show that r_i is is an isomorphism of surface groups between S_i and one of the groups R_j. The surfaces Σ_i and Ξ_j are thus isomorphic, in particular they have the same number of boundary components: by connectedness of Σ, this implies that $\Sigma = \Sigma_i \cup \Xi_j$. The graph of group Γ corresponding to the hyperbolic floor (S, R, r) is a graph on two vertices only, and $i = 1$ and $j = 0$. Also, Σ_1 is the closure of the complement of Ξ_0 in Σ. Now we have $\chi(\Xi_0) = \chi(\Sigma_1)$ and $\chi(\Sigma) = \chi(\Xi_0) + \chi(\Sigma_1)$ so that $\chi(\Xi_0) = \chi(\Sigma)/2$. We see that Ξ_0 is a minor subsurface, and Proposition 5.20 is proved in this particular case.

If none of the maps r_i are non-pinching, the idea is to write r_1 as $r'_1 \circ \rho$, where r'_1 is non-pinching with respect to some suitable surface groups contained in $\rho(S_1)$.

Definition 5.23: (essential set of curves pinched by f) *Let S_1 be the fundamental group of a surface Σ_1. Let $f : S_1 \to G$ be a group morphism whose restriction to each boundary subgroup is injective.*

If the elements of S_1 corresponding to a simple closed curve γ on Σ_1 lie in the kernel of f, we say γ is pinched by f. Let $\mathcal{C}$ be an essential set of curves. If all its elements are pinched by f, we say it is an essential set of curves pinched by f, and if it is maximal among such sets, we say it is a maximal essential set of curves pinched by f.

Recall that we consider essential sets of simple closed curves up to homotopy.

Definition 5.24: (pinching map $\rho_{\mathcal{C}}$) *Let S_1 be the fundamental group of a surface Σ_1. Let $\mathcal{C}$ be an essential set of curves on Σ_1. Denote by $N(\mathcal{C})$ the subgroup of S_1 normally generated by elements corresponding to the curves of $\mathcal{C}$. The pinching map $\rho_{\mathcal{C}}$ is the quotient map $S_1 \to S_1/N(\mathcal{C})$.*

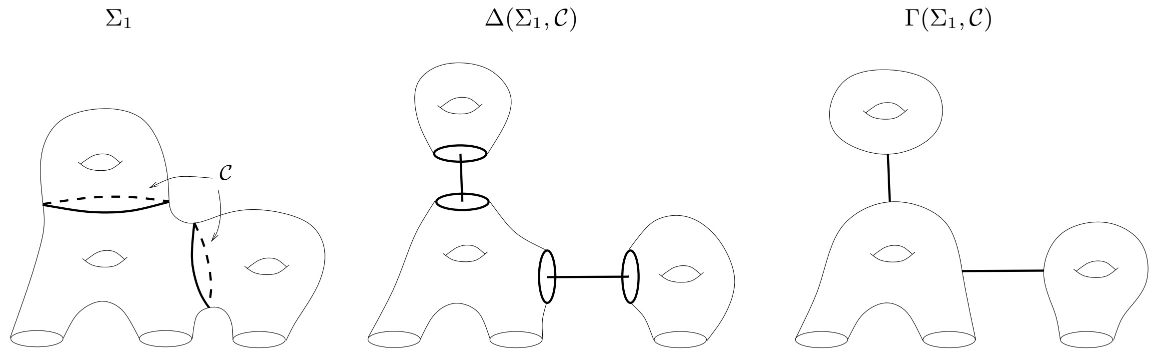

Figure 3: The construction of the graph of groups $\Gamma(\Sigma_1, \mathcal{C})$.

The pinching map $\rho_{\mathcal{C}}$ is injective on each boundary subgroup. If $\Delta(\Sigma_1, \mathcal{C})$ is the graph of groups decomposition of S_1 dual to $\mathcal{C}$, its edge groups have trivial image by $\rho_{\mathcal{C}}$, and its vertex group are quotiented by some of their boundary subgroups.

Definition 5.25: (graph of groups $\Gamma(S_1, \mathcal{C})$) *Let S_1 be the fundamental group of a surface Σ_1. Let $\mathcal{C}$ be an essential set of curves on Σ_1. We denote $\Gamma(\Sigma_1, \mathcal{C})$ the graph of group decomposition of $\rho_{\mathcal{C}}(S)$ obtained by replacing, in the decomposition $\Delta(\Sigma_1, \mathcal{C})$ of S dual to $\mathcal{C}$, each vertex and edge group by its image by $\rho_{\mathcal{C}}$.*

Note that a vertex group S_1^{Δ} of $\Delta(\Sigma_1, \mathcal{C})$ is the fundamental group of a subsurface Σ_1^{Δ} of Σ_1. The image of S_1^{Δ} by $\rho_{\mathcal{C}}$ is the fundamental group of the surface Σ_1^{Γ} obtained by gluing discs to the boundary components of Σ_1^{Δ} corresponding to curves of $\mathcal{C}$ (see Figure 3). If all the boundary components of Σ_1^{Δ} correspond to curves of $\mathcal{C}$, the image of S_1^{Δ} by $\rho_{\mathcal{C}}$ is the fundamental group of a closed surface. Then, we call the corresponding vertex group S_1^{Γ} of $\Gamma(\Sigma_1, \mathcal{C})$ an interior vertex group. A non-interior vertex group, on the other hand, is the fundamental group of a surface with boundary.

Remark 5.26: *If Σ_1^{Γ} is a surface corresponding to a vertex of $\Gamma(\Sigma_1, \mathcal{C})$, then $\chi(\Sigma_1^{\Gamma})$ is at least $\chi(\Sigma_1)$, with equality if and only if $\mathcal{C}$ is empty.*

Remark 5.27: *If $f : S_1 \to G$ is a morphism from the fundamental group of a surface with boundary Σ_1 to a group G which is injective on boundary subgroups, and if $\mathcal{C}$ is a maximal essential set of curves pinched by f, the map f factors as $f' \circ \rho_{\mathcal{C}}$, and f' is non-pinching with respect to the surfaces corresponding to vertices of $\Gamma(S_1, \mathcal{C})$.*

We can now prove Proposition 5.20.

Proof of Proposition 5.20. Suppose H is a free factor of a minor subsurface Σ_0 of Σ, and denote by S_0 the fundamental group of Σ_0. It is straightforward to see that there is an essential set of curves $\mathcal{C}$ on the closure Σ_1 of $\Sigma - \Sigma_0$ such that $\Gamma(\Sigma_1, \mathcal{C})$ has a unique non-interior surface Σ_1^{Γ}, and that this surface is homeomorphic to Σ_0 via some homeomorphism ϕ which restricts to the identity on $\partial\Sigma_1$. Let $r' : \rho_{\mathcal{C}}(S) \to S_0$ be the identity on S_0, the map ϕ_* on the vertex group S_1^{Γ} corresponding to Σ_1^{Γ}, and the trivial map on the other vertex groups of $\Gamma(\Sigma_1, \mathcal{C})$. It is easy to check that $r = r' \circ \rho_{\mathcal{C}}$ is a well-defined retraction $S \to S_0$ which makes (S, S_0, r) a hyperbolic floor. Then S is a hyperbolic tower over H.

Conversely, suppose now that S is a hyperbolic tower over H. Since S is freely indecomposable, the tower has at least one floor, so S admits a structure of hyperbolic floor (S, R, r) with corresponding graph of groups decomposition Λ. We know that H is elliptic in Λ and that it is contained in R, so we can assume that it is contained in a non surface type vertex stabiliser R_0 of T_{Λ}. Let S_1 be a surface type vertex stabiliser of T_{Λ} adjacent to R_0.

By Theorem III.2.6 of [MS84], Λ is the splitting dual to a set of simple closed curves on Σ. Denote by Σ_0 and Σ_1 the subsurfaces of Σ corresponding to R_0 and S_1 respectively, so that both R_0 and S_1

are endowed with a structure of surface group with boundary. Note that then, the intersection of R_0 and S_1 is an infinite cyclic subgroup Z which is a maximal boundary subgroup of S_1 and R_0.

We want to consider the restriction r_1 of r to S_1: it is a morphism from S_1 to R, and it restricts to the identity on Z. Let $\mathcal{C}$ be a maximal essential set of curves on Σ_1 pinched by r_1. The map r_1 factors as $r'_1 \circ \rho_{\mathcal{C}}$. Let S_1^Γ be the non-interior surface type vertex group of $\Gamma(S_1, \mathcal{C})$ containing Z, denote by Σ_1^Γ the corresponding surface. By Remark 5.27, $r'_1|_{S_1^\Gamma}$ is non-pinching with respect to Σ_1^Γ. Now by an argument similar to that in Remark 5.21, the image of S'_1 by r'_1 is contained in a non surface type vertex groups of T_Γ: clearly, this group must be R_0. Moreover, r'_1 sends non-conjugate maximal boundary elements of S'_1 to non-conjugate maximal boundary elements of R_0. We can thus apply Lemma 5.22 to conclude that $r'_1|_{S_1^\Gamma}$ is an isomorphism of surface groups with boundary between S_1^Γ and R_0, so that Σ_0 and Σ_1^Γ are homeomorphic.

In particular, Σ_0 and Σ_1^Γ have the same number of boundary components. This implies that the complement of Σ_0 in Σ is connected, so it is exactly Σ_1. Since Σ_0 and Σ_1^Γ are homeomorphic, we have $\chi(\Sigma_0) = \chi(\Sigma_1^\Gamma)$. On the other hand, by Remark 5.26, we have $\chi(\Sigma_1^\Gamma) \geq \chi(\Sigma_1)$, with equality if and only if $\mathcal{C}$ is empty, in which case we have $\Sigma_1^\Gamma = \Sigma_1$ and Σ_0 and Σ_1 are homeomorphic. Now we have $\chi(\Sigma) = \chi(\Sigma_0) + \chi(\Sigma_1)$. Thus we get $\chi(\Sigma_0) \geq \chi(\Sigma)/2$, and if we have equality, Σ_0 and Σ_1 are homeomorphic. Finally, if Σ_0 is non-orientable, so is Σ_1^Γ and thus so is Σ_1. Thus Σ_0 is a minor subsurface of Σ. □

6 A property of JSJ-like decompositions

To complete the proof of 1.2, we now need to prove Propositions 5.11 and 5.12. This will be done in Section 8, using the results that we will expose now and in Section 7.

In this section, we show that if a preretraction $G \to G$ relative to some cyclic JSJ-like decomposition of G satisfies some strong injectivity conditions on the vertex groups, it must be an isomorphism.

Proposition 6.1: *Let G be a torsion-free hyperbolic group, and let Λ be a JSJ-like decomposition of G. Let $\theta : G \to G$ be a preretraction with respect to Λ, which sends surface type vertex groups of Λ isomorphically to conjugates of themselves. Then θ is an isomorphism.*

Proof. First note that if G is cyclic, the only JSJ-like decomposition it admits is the trivial one, for which the result is immediate. We may thus assume that G is not cyclic.

Denote by T the Bass-Serre tree T_Λ corresponding to Λ. To prove the proposition, we will construct a bijective simplicial map $j : T \to T$, such that j is equivariant with respect to θ, i.e. so that for any element g of G and any vertex v of T we have:

$$j(g \cdot v) = \theta(g) \cdot j(v).$$

The stabilisers of an edge e of T and of a vertex v of T in the standard action of G on T are denoted by G_e and G_v respectively.

1. Construction of the map j on vertices. By hypothesis, for each vertex v of T, there is an element g_v of G such that $\theta(G_v) = g_v G_v g_v^{-1}$. We set the image of v by j to be $g_v \cdot v$. Its stabiliser is exactly $\theta(G_v)$, and recall that distinct vertices of the tree corresponding to a JSJ-like decomposition have distinct stabilisers, so this property defines $j(v)$ uniquely. Thus the image of $g \cdot v$ by j is the unique vertex whose stabiliser is $\theta(g)\theta(G_v)\theta(g^{-1})$, namely $\theta(g) \cdot j(v)$, and the map $v \mapsto j(v)$ is equivariant with respect to θ. Note that $j(v)$ is in the orbit of v, and thus is of the same type. Note also that $G_{j(v)} = \theta(G_v) \simeq G_v$.

2. The map $v \mapsto j(v)$ can be extended to a simplicial map $j : T \to T$. We need to check that adjacent vertices are sent on adjacent vertices. Suppose v, w adjacent: without loss of generality G_v is not a surface type vertex group. The intersection $G_v \cap G_w$ is an infinite cyclic group. On G_v, the map θ is just conjugation by the element g_v of G, so if we let $C := \theta(G_v \cap G_w)$, we have $C = g_v(G_v \cap G_w)g_v^{-1}$.

In particular, C is non-trivial. Moreover, C fixes both $j(v)$ and $j(w)$, thus $j(v)$ and $j(w)$ are at a distance at most 2 by 2-acylindricity. We will first show that it cannot be 2, then that it cannot be 0.

- Assume the distance is 2. The vertex u between $j(v)$ and $j(w)$ is a Z type vertex, which implies in particular that $j(v)$ and $j(w)$, and thus v and w, are not Z type vertex. Note that C is contained in $g_v G_w g_v^{-1}$, so it fixes the vertex $g_v \cdot w$. This vertex is at a distance 1 from $j(v)$, thus it is distinct from $j(v)$ and from $j(w)$. Its stabiliser is not cyclic, thus it is distinct from u. Hence C stabilises points $j(w)$ and $g \cdot w$ which are at a distance 3 of each other. This is a contradiction.

- Assume now $j(v) = j(w)$. Thus v and w are in the same orbit (in particular they must be of rigid type, since they are adjacent). Let $a \in G$ be such that $w = a \cdot v$. We have $G_w = aG_v a^{-1}$. We see that $\theta(a) \in \theta(G_v)$, since $j(v) = j(w) = j(a \cdot v) = \theta(a) \cdot j(v)$ and the stabiliser of $j(v)$ is $\theta(G_v)$. Thus there exists $a' \in G_v$ such that $\theta(a') = \theta(a)$.

  Let $C_1 := G_v \cap G_w$, i.e. C_1 is the stabiliser of the edge e between v and w. Let $C_2 \leq G_v$ be such that $C_1 = aC_2a^{-1}$. Let c_1 generate C_1, and $c_2 := a^{-1}c_1 a$ generate C_2. Note that by Remark 5.8, C_1 is maximal abelian in G since it is the stabiliser of an edge which connects two rigid vertices. Now $\theta(c_2) = \theta(a^{-1})\theta(c_1)\theta(a)$ so that $\theta(c_2) = \theta(a'^{-1}c_1 a')$. By injectivity of θ on G_v, $c_2 = a'^{-1}c_1 a'$. Thus $a'a^{-1}$ commutes with c_1, so it must be in C_1 and thus in G_v. But $a' \in G_v$ so we deduce $a \in G_v$ and $G_w = aG_v a^{-1} = G_v$. Since distinct vertices have distinct stabilisers, we get a contradiction.

Thus we can extend $v \mapsto j(v)$ to a simplicial map $j : T \to T$.

3. Injectivity of j. It is enough to show that there are no foldings, i.e. that no two edges adjacent to a same vertex are sent to the same edge by j. Suppose that two vertices w, w' of T are adjacent to a vertex v, and that the edges $e = [v, w]$ and $e' = [v, w']$ are sent on a same image by j. Let g_e be a generator of the stabiliser G_e of e, and $g_{e'}$ a generator of the stabiliser $G_{e'}$ of e'.

Let us see that v must be a Z type vertex. We know that the stabiliser of $j(e)$ contains $\theta(g_e)$ and $\theta(g_{e'})$, so that $\theta([g_e, g_{e'}]) = 1$. As θ is injective on G_v, the elements g_e and $g_{e'}$ of G_v commute. Thus they have a common power which fixes both e and e': by strong 2-acylindricity, v is a Z type vertex. This implies that w, w', and $j(w)$ are not type Z vertices.

It is clear that G_w and $G_{w'}$ must be conjugate since $j(w) = j(w')$, so w and w' are in the same orbit. Let γ be an element of G such that $w' = \gamma \cdot w$. Note that γ does not lie in G_w. However, there is an element a of G_w such that $\theta(a) = \theta(\gamma)$. Indeed, $\theta(\gamma) \cdot j(w) = j(\gamma \cdot w) = j(w') = j(w)$ so that $\theta(\gamma)$ stabilises $j(w)$ thus lies in $\theta(G_w)$.

Let g be an element of G_v which stabilise both e and e': then g is both in G_w and in $\gamma G_w \gamma^{-1}$. Let g' be an element of G_w be such that $g = \gamma g' \gamma^{-1}$. We have

$$\begin{aligned} \theta(g) &= \theta(\gamma)\theta(g')\theta(\gamma^{-1}) \\ &= \theta(a)\theta(g')\theta(a^{-1}) = \theta(ag'a^{-1}). \end{aligned}$$

Since θ is injective on G_w, we deduce that $g = ag'a^{-1}$ so $g' = \gamma^{-1}g\gamma = a^{-1}ga$. This shows $[\gamma a^{-1}, g] = 1$, so γa^{-1} preserves the set $\mathrm{Fix}(g)$ of fixed point of g. But $\mathrm{Fix}(g)$ has diameter 2 and is centred on v, so γa^{-1} fixes v, and γa^{-1} lies in G_v. Now a was chosen so that $\theta(\gamma) = \theta(a)$, so $\theta(\gamma a^{-1}) = 1$. By injectivity of θ on G_v, we get $\gamma = a$. This is a contradiction since γ is not in G_w, but a is.

4. Injectivity of θ. The injectivity of j implies the injectivity of θ.

5. Surjectivity of j. We prove this by showing that if a vertex v is in the image of j, all the edges adjacent to v are also in the image. Suppose v is in the image of j, there exists an element g^v of G such that $j(g^v \cdot v) = v$. Pick $e_1, \ldots, e_r$ some representatives of the orbits of edges adjacent to v. The image e'_k of $g^v \cdot e_k$ by j must be adjacent to v.

We claim that if e_k and e_l lie in different orbits, so do e'_k and e'_l. Indeed, if e'_k and e'_l are in the same orbit, there exists α in G_v such that $\alpha \cdot e'_k = e'_l$. Since the action has no inversions, α must fix v. As v is in the image of j, its stabiliser is in the image of θ so there exists an element a of G such that $\theta(a) = \alpha$. Thus $\theta(a) \cdot j(g^v \cdot e_k) = j(g^v \cdot e_l)$, so by equivariance of j we get $j(ag^v \cdot e_k) = j(g^v \cdot e_l)$. By injectivity of j this means e_k and e_l are in the same orbit: this proves the claim. Thus the edges e'_k form a system of representative of the orbits of edges adjacent to v.

Now let e be an edge adjacent to v: there is an edge e'_k which is in the orbit of e, thus there is an element β of G such that $\beta \cdot e'_k = e$. Since the action has no inversions, β must fix v. We know G_v is in the image of θ so $\beta = \theta(b)$ for some b. Thus $j(b \cdot (g^v \cdot e_k)) = \theta(b) \cdot j(g^v \cdot e_k) = \beta \cdot e'_k = e$, so e is in the image of j. Hence all the vertices which neighbour v are in the image of j. This local surjectivity condition implies global surjectivity of j.

6. Surjectivity of θ. Let g be an element of G, and let v be a vertex of T with non-cyclic stabiliser. By surjectivity of j there exists w such that $j(w) = v$, and w' such that $j(w') = g \cdot v$. Clearly w and w' are in the same orbit. Thus $G_{w'} = hG_wh^{-1}$ for some h. We see that

$$gG_vg^{-1} = G_{g\cdot v} = \theta(G_{w'}) = \theta(h)\theta(G_w)\theta(h^{-1}) = \theta(h)G_v\theta(h^{-1}).$$

We get $G_v = g^{-1}\theta(h)G_v\theta(h)^{-1}g$. Thus G_v stabilises both v and $g^{-1}\theta(h) \cdot v$. Since G_v is not cyclic, $v = g^{-1}\theta(h) \cdot v$ so $g^{-1}\theta(h)$ is in G_v. Since we know that G_v is in the image of θ, we get that g is in the image of θ.

We proved that θ is bijective, this terminates the proof. □

7 Non-pinching maps and the finite index property

Recall that in Section 3, we saw that a morphism $f : S \to S'$ between surface groups with boundary which sends boundary elements to boundary elements and is non-pinching has image either contained in a boundary component, or with finite index in S' (Lemma 3.10). Moreover, we saw that in the second case, the surface Σ corresponding to S must have complexity greater than that of the surface Σ' corresponding to S' (Lemma 3.12).

In the first part of this section, we show a result which can be thought of as a generalisation of this to morphisms $f : A \to G$ between fundamental groups of graphs of groups with surfaces Λ and Γ. In particular, we give conditions under which any given surface type vertex group of Γ intersect the image of the morphism f either in a boundary subgroup, or in a subgroup of finite index. This is what we call the finite index property. The analogue of the 'boundary elements sent to boundary elements' condition will be the assumption that f sends non-surface type vertex groups and edge groups to non-surface type vertex groups and edge groups respectively. And, as in Lemma 3.10, we will need a 'non-pinching' condition on the map f, that is, we'll assume that its kernel does not contain elements corresponding to simple closed curves on the surfaces of Λ. We will also show what the existence of such a non-pinching map implies on the complexities of the surfaces of Λ and Γ.

In the second half of the section, we try and generalise these results to morphisms $f : A_1 * \ldots * A_l \to G$, where each of the A_i and G admit a decomposition as graphs of groups with surfaces and with infinite cyclic edge groups.

7.1 Graphs of groups with surfaces

7.1.1 Elliptic refinements of graphs of groups with surfaces

Let A and G be fundamental groups of graphs of groups with surfaces Λ and Γ respectively. Let $f : A \to G$ be a morphism which sends edge groups and non surface type vertex groups of Λ into non surface type vertex groups of Γ.

Each surface type vertex group S of Λ corresponding to a surface Σ acts on the tree T_Γ corresponding to Γ via the map f, and boundary subgroups of S are elliptic in this action. By Lemma 3.5, there

exists an essential set of curves $\mathcal{C}^+(\Sigma)$ on Σ, such that the image by f of any vertex group of the graph of groups $\Delta(\Sigma, \mathcal{C}^+(\Sigma))$ dual to $\mathcal{C}^+(\Sigma)$ is elliptic in T_Γ. We can then refine Λ by the graphs of groups $\Delta(\Sigma, \mathcal{C}^+(\Sigma))$: every vertex group of the refined graph of groups Λ^+ thus obtained is elliptic in the action of A on T_Γ via f. Denote by $\mathcal{C}^+$ the union of all the sets $\mathcal{C}^+(\Sigma)$.

Definition 7.1: (elliptic refinement of a graph of group) *We call the graph of groups Λ^+ built as above an elliptic refinement of Λ relative to f and Γ, given by the set of curves $\mathcal{C}^+$.*

Remark 7.2: *There exists a (not necessarily simplicial) map $t^+ : T_{\Lambda^+} \to T_\Gamma$ which is equivariant with respect to the action of A on T_{Λ^+} on one hand, and the action of A on T_Γ via f on the other hand. Indeed, the image by f of any vertex groups of Λ^+ is elliptic in T_Γ, so we can send each vertex of T_{Λ^+} on a vertex of T_Γ stabilised by the image of its stabiliser, and extend equivariantly. Moreover, by some straightforward modifications, the map can be made locally minimal in the sense of Definition 3.3.*

7.1.2 Non-pinching maps on graphs of groups with surfaces

Definition 7.3: (non-pinching with respect to a graph of groups with surfaces) *Let Λ_1 be a graph of groups with surfaces with fundamental group A_1. We say that a morphism $f : A_1 \to A$ is non-pinching with respect to Λ_1 if the restriction of f to each surface type vertex group of Λ_1 is non-pinching.*

Setting. For the rest of Section 7.1, A_1 and A are groups which admit decompositions Λ_1 and Λ as graphs of groups with surfaces whose edge groups are infinite cyclic. Also, $f : A_1 \to A$ is a morphism which sends non surface type vertex groups and edge groups of Λ_1 injectively into non surface type vertex groups and edge groups of Λ respectively.

By the previous subsection, we can define an elliptic refinement Λ_1^+ of Λ_1 with respect to f and Λ. We then know that there exists an f-equivariant locally minimal map $t^+ : T_{\Lambda_1^+} \to T_\Lambda$.

Lemma 7.4: *Suppose we are in the setting above. If f is non-pinching with respect to Λ_1, for any surface type vertex v of T_Λ with stabiliser S the following are equivalent*

(i) v lies in the image of $T_{\Lambda_1^+}$ by t^+;

(ii) there is a conjugate of a surface type vertex group S^+ of Λ_1^+ whose image by f lies in S;

(iii) there is a conjugate of a surface type vertex group S^+ of Λ_1^+ whose image by f is a subgroup of finite index of S;

(iv) the intersection of S with $f(A_1)$ is not contained in a boundary subgroup of S.

Before giving a proof, we make the following useful remark

Remark 7.5: *Let G be the fundamental group of a graph of groups with surfaces Λ. If S is a surface type vertex group, it is free so maximal cyclic subgroups are malnormal in S. This implies that the tree corresponding to Λ is 1-acylindrical next to surface type vertices, that is, no non-trivial element stabilises two distinct adjacent edges whose common endpoint is a surface type vertex.*

Proof. The fact that f is non-pinching on Λ_1, injective on its edge groups, and that the edge groups of Λ are infinite cyclic implies that it is also non-pinching on Λ_1^+ and injective on its edge groups.

$(i) \Rightarrow (ii)$: If w is a non surface type vertex of $T_{\Lambda_1^+}$ with non-cyclic stabiliser R, then f is injective on R so $f(R)$ is non-cyclic, thus it stabilises exactly one vertex in T_Λ. But $f(R)$ lies in a non surface type vertex group of Λ, so $t^+(w) \neq v$.

Suppose now that w is a non surface type vertex w of $T_{\Lambda_1^+}$ with cyclic stabiliser Z. By local minimality of t^+, either the image of an open neighbourhood of w intersects at least two edges adjacent to $t^+(w)$, or $f(Z)$ properly contains all the edge group of the unique edge on which an open neighbourhood of w is sent. In the first case, note that two edges of T_Γ adjacent to $t^+(w)$ are stabilised by a same non-trivial element, so by 1-acylindricity next to surface type vertices, $t^+(w)$ is not of surface

type. In the second case, note that edge groups adjacent to surface type vertices are maximal cyclic in the surface group, so $t^+(w)$ can not be of surface type.

Finally, if e is an edge of $T_{\Lambda_1^+}$, the image of its interior is stabilised by a non-trivial element, thus if this image is not constant, it does not contain any surface type vertices by 1-acylindricity next to surface type vertices and local minimality of t^+.

Thus we see that if a surface type vertex with stabiliser S is in the image of t^+, it means that it is the image of some surface type vertex of $T_{\Lambda_1^+}$ with stabiliser S^+. Thus $f(S^+) \leq S$ as claimed.

$(ii) \Rightarrow (iii)$: The map f sends edge groups of $T_{\Lambda_1^+}$ to edge groups of T_Λ, thus boundary subgroups of S^+ are sent to boundary subgroups of S. Moreover, by local minimality of t^+ and 1-acylindricity next to surface type vertices, $f(S^+)$ is not contained in a boundary subgroup of S. By Lemma 3.10, this means that $f(S^+)$ has finite index in S.

$(iii) \Rightarrow (iv)$: This is clear.

$(iv) \Rightarrow (i)$: If v lies outside of $t^+(T_{\Lambda_1^+})$, the intersection between $f(A_1)$ and S stabilises both v and $t^+(T_{\Lambda_1^+})$, thus it stabilises the non-trivial path between them. Thus it stabilises one of the edges adjacent to v, which implies that it is contained in a boundary subgroup of S. □

7.1.3 Surface complexity of graphs of groups

Definition 7.6: (complexity of a set of surfaces, surface complexity of a graph of groups with surfaces) *Let $\mathcal{S} = \{\Sigma_i \mid 1 \leq i \leq l\}$ be a set of surfaces with boundary, and recall that $k(\Sigma_i)$ denotes the topological complexity of Σ_i given by Definition 3.1. The complexity $K(\mathcal{S})$ is the finite sequence $(k(\Sigma_i))_{1 \leq i \leq l}$ of the complexities of surfaces of $\mathcal{S}$ arranged in decreasing order.*

We order the complexities of sets of surfaces lexicographically, that is

$$k(\Sigma_1) \dots k(\Sigma_l) < k(\Sigma'_1) \dots k(\Sigma'_{l'})$$

if $\{i \mid k(\Sigma_i) \neq k(\Sigma'_i); 1 \leq i \leq \min\{l, l'\}\}$ is non-empty, has minimum j, and $k(\Sigma_j) < k(\Sigma'_j)$; or if the set is empty and $l < l'$.

If Λ is a graph of groups with surfaces, its surface complexity is the complexity of its set of surfaces.

The proof of the following lemma is a straightforward exercise in computing Euler characteristic.

Lemma 7.7: *If $\mathcal{C}^+$ is a non-empty set of curves on the surfaces of a graph of groups with surfaces Λ_1, the surface complexity of an elliptic refinement Λ_1^+ of Λ_1 given by $\mathcal{C}^+$ is strictly smaller than that of Λ_1.*

The following lemma gives us a relation between the surface complexities of the graphs of groups Λ_1 and Λ when the map $f : A_1 \to A$ is non-pinching.

Lemma 7.8: *Suppose we are in the setting given above. If f is non-pinching with respect to Λ_1, and if $t^+ : T_{\Lambda_1^+} \to T_\Lambda$ is surjective, the surface complexity of Λ_1 is at least that of Λ.*

Proof. Each surface type vertex lies in the image of t^+, so by Lemma 7.4, for each surface type vertex group S of Λ there is a surface type vertex group S^+ of Λ_1^+ such that $f(S^+)$ has finite index in a conjugate of S. By Lemma 3.12, the complexity of the surface corresponding to S^+ is greater than that of the surface corresponding to S. In this way, to each surface of Λ corresponds a surface of Λ_1^+ whose complexity is greater, and this correspondence gives an injection from the set of surfaces of Λ to the set of surfaces of Λ_1^+. This implies that the surface complexity of Λ is smaller than that of Λ_1^+, which in turn is smaller than that of Λ_1 by Lemma 7.7. □

7.2 Finite index property for free products

We now want to prove a proposition that should be thought of as a generalisation of Lemma 7.4 in the case where instead of a morphism from A_1 to A, we have a map from a free product $A_1 * \dots * A_l$ to A. We will see that up to conjugation on these free factors, we still control which surface type vertex groups of Λ intersect the image of a non-pinching map in more than a boundary subgroup.

Proposition 7.9: *Let $A_1, \dots, A_l$ be groups which admit JSJ-like decompositions $\Lambda_1, \dots, \Lambda_l$ and let Λ be a graph of groups with surfaces with fundamental group A. Assume that for each i, the surface complexity $K(\Lambda_i)$ of Λ_i is strictly smaller than the surface complexity $K(\Lambda)$ of Λ.*

Suppose $h : A_1 * \dots * A_l \to A$ is a map which sends non surface type vertex groups and edge groups of the graphs of groups Λ_i injectively into non surface type vertex groups and edge groups of Λ respectively, and such that the maps $h|_{A_i}$ are non-pinching with respect to the graphs Λ_i. For each i with $1 \leq i \leq l$, let Λ_i^+ be an elliptic refinement of Λ_i with respect to $h|_{A_i}$ and Λ.

Then there exists a map $\tilde{h} : A_1 * \dots * A_l \to A$ which satisfies the following properties:

1. *$\tilde{h}|_{A_i}$ coincides with $h|_{A_i}$ up to conjugation;*
2. *$\tilde{h}(A_1 * \dots * A_l) = \tilde{h}(A_1) * \dots * \tilde{h}(A_l)$;*
3. *for any surface type vertex group S of Λ, the following are equivalent:*
   - *(i) The intersection of S with $\tilde{h}(A_1 * \dots * A_l)$ is not contained in a boundary subgroup of S.*
   - *(ii) There is a conjugate in $A_1 * \dots * A_l$ of a surface type vertex group S^+ of one of the graphs of groups Λ_i^+ whose image by $\tilde{h}$ has finite index in S.*

The reason why we might need to replace each $h|_{A_i}$ by a conjugate, is that there might be some surface group S of Λ which intersects several of the $h(A_i)$ in a boundary component. Then the intersection of S with $h(A_1 * \dots * A_l)$ might be neither a finite index subgroup, nor a boundary subgroup of S.

To prove Proposition 7.9 we will need some lemmas about actions on trees. Recall that a G-tree is said to be minimal if it doesn't contain any proper G-invariant subtree, and that any G-tree contains a unique minimal G-subtree. A G-tree is said to be irreducible if none of its ends is fixed by G. Finally, a G-tree is k-acylindrical if the diameter of the set $\mathrm{Fix}(g)$ of fixed points of an element g of G is at most k.

Lemma 7.10: *Let G be a finitely generated group, and let T be a minimal irreducible G-tree. If τ and τ' are proper subtrees of T, for any integer D there is a translate of τ' by an element of G which lies at a distance at least D of τ.*

Proof. By Lemma 4.3 in [Pau89], the hypotheses allow us, for any two distinct vertices v and w of T, to find an element of G which is hyperbolic in the action of G on T and whose axis contains the path between v and w.

Suppose first that the smallest tree τ_0 containing $\tau \cup \tau'$ is a proper subtree of T. Let K be a connected component of the complement of τ_0 in T, and let u be the vertex of T such that $\overline{K} \cap \tau_0 = \{u\}$. By minimality and irreducibility of the action, K is not a line, so we can find points v and w in such a component such that the tripod formed by v, w, and u is non-trivial. We pick a hyperbolic element g whose axis contains the path between v and w. The projection of τ and τ' on the axis of g is reduced to a point. Thus $g^D \cdot \tau'$ is at distance greater than D of τ.

If on the other hand $\tau_0 = T$, we pick vertices v, w of the tree which are in τ' but not in τ, and in τ but not in τ' respectively. Now τ lies in the connected component of $T - \{v\}$ containing w and τ' lies in the connected component of $T - \{w\}$ containing v. Thus the intersection $\tau \cap \tau'$ lies in the connected component of $T - \{v, w\}$ containing the arc between v and w. Pick a hyperbolic element whose axis contains the path between v and w. By applying a suitable power of this element we can translate τ' away from τ. □

Lemma 7.11: *Let G be a finitely generated group, and let τ be a k-acylindrical minimal G-tree. Suppose G_1 and G_2 are subgroups of G which generate G, and whose minimal subtrees T_1 and T_2 in τ lie at a distance at least $2k+3$ from each other. Then for any vertex v of τ*

- *either* $\mathrm{Stab}_G(v)$ *stabilises an edge adjacent to v;*
- *or v lies in a translate of T_i by an element of G, and in this case* $\mathrm{Stab}_G(v)$ *stabilises this translate.*

Moreover, we have $G = G_1 * G_2$.

Proof. Denote by D the path joining T_1 to T_2. The tree τ is the union of translates of T_1, T_2 and D by elements of G. Let $\hat{T}_i$ for $i = 1, 2$ be the set of points whose distance to T_i is at most $k + 1$: note that $\hat{T}_1$ and $\hat{T}_2$ are disjoint. Denote by $\hat{D}$ the subsegment of D which joins $\hat{T}_1$ and $\hat{T}_2$. Let B_i be the complement in $\tau - \hat{T}_i$ of the connected component containing the interior of $\hat{D}$ for $i = 1, 2$.

By k-acylindricity, an element of G_1 sends points of $\hat{D}$, of $\hat{T}_2$ and of B_2 into B_1, and an element of G_2 sends points of $\hat{D}$, of $\hat{T}_1$ and of B_1 into B_2.

If $v \in \hat{D}$, its image by a non-trivial element of G lies in $B_1 \cup B_2$, thus $\mathrm{Stab}_G(v)$ is trivial. In particular, if u is a non trivial word in G_1 and G_2, then $u \cdot v$ is distinct from v, so u represents a non-trivial element of G. Thus $G = G_1 * G_2$.

If v lies in $\hat{T}_1$ and $g \cdot v = v$ then g lies in G_1: indeed, otherwise we can see that $g \cdot v$ lies in $B_1 \cup B_2$. Thus if v lies in T_1, the stabiliser of v also stabilises T_1, and if v lies in $\hat{T}_1 - T_1$, the stabiliser of v also stabilises the path between v and T_1, so it stabilises an edge adjacent to v. We get a similar result if v lies in $\hat{T}_2$. If v lies in a translate $g \cdot \hat{D}$ of $\hat{D}$, or in a translate $g \cdot \hat{T}_i$ of $\hat{T}_i$, we apply the results above to $g^{-1} \cdot v$. $\square$

We can now prove Proposition 7.9.

Proof. We prove by induction on n that we can find $\tilde{h} : A_1 * \ldots * A_n \to A$ for which properties 1, 2 and 3 hold, as well as

4. the minimal subtree of $\tilde{h}(A_1 * \ldots * A_n)$ in T_Λ is a proper subtree.

For $n = 1$, if we take $\tilde{h} = h$ the result holds by Proposition 7.4. Since we assumed that $K(\Lambda_1) < K(\Lambda)$, the minimal subtree of $h(A_1)$ does not cover T_Λ by Lemma 7.8.

Suppose that the induction hypothesis holds for $n - 1$. Let h be a map from $A_1 * \ldots * A_n$ to A which satisfies all the hypotheses. The induction hypothesis gives us a map $\tilde{h}$ from $A_1 * \ldots * A_{n-1}$ to A such that $\tilde{h}|_{A_i}$ coincides with $h|_{A_i}$ up to conjugation for $i < n$, and such that the minimal subtree T_1 of $G_1 = \tilde{h}(A_1 * \ldots * A_{n-1})$ is a proper subtree of T_Λ.

Consider the minimal tree of $h(A_n)$: since we assumed $K(\Lambda_n) < K(\Lambda)$, by Lemma 7.8, it is also a proper subtree of T_Λ. Thus by Lemma 7.10, it has a translate T_2 by an element α of A which lies at a distance at least 7 of T_1. Extend $\tilde{h}$ to A_n by setting $\tilde{h}|_{A_n} = \mathrm{Conj}(\alpha) \circ h|_{A_n}$. Then T_2 is the minimal subtree of $G_2 = \tilde{h}(A_n)$.

The map $\tilde{h}$ thus defined clearly satisfies 1.

Note that the group G generated by $G_1 = \tilde{h}(A_1 * \ldots * A_{n-1})$ and $G_2 = \tilde{h}(A_n)$ is precisely $\tilde{h}(A_1 * \ldots * A_n)$. If we denote its minimal subtree by τ the hypotheses of Lemma 7.11 are satisfied so $\tilde{h}(A_1 * \ldots * A_n) = \tilde{h}(A_1 * \ldots * A_{n-1}) * \tilde{h}(A_n)$. By induction hypothesis we get

$$\tilde{h}(A_1 * \ldots * A_n) = \tilde{h}(A_1) * \ldots * \tilde{h}(A_l)$$

so that $\tilde{h}$ satisfies 2. Moreover, τ is properly contained in T_Λ, since the points which lie on the path between T_1 and T_2 are branching points in T_Λ, but not in τ, and property 4 holds.

Now let v be a surface type vertex of T_Λ, and denote by S its stabiliser. If v lies outside of τ, the intersection $S \cap \tilde{h}(A_1 * \ldots * A_n)$ stabilises both v and τ, thus it is a boundary subgroup of S. We may thus assume that v lies in τ. By Lemma 7.11, either $S \cap \tilde{h}(A_1 * \ldots * A_n)$ is contained in the stabiliser of an edge adjacent to v, so it is a boundary subgroup of S, or v lies in a translate of T_1 or T_2.

If v lies in T_1 itself, Lemma 7.11 also tells us that $\mathrm{Stab}_G(v)$, i.e. the intersection $S \cap \tilde{h}(A_1 * \ldots * A_n)$, is contained in the stabiliser G_1 of T_1, namely $\tilde{h}(A_1 * \ldots * A_{n-1})$. By induction hypothesis we have two possibilities: either the intersection $S \cap \tilde{h}(A_1 * \ldots * A_{n-1})$ lies in a boundary subgroup of S, but then so does the intersection $S \cap \tilde{h}(A_1 * \ldots * A_n)$; or there is a conjugate of a surface type vertex group S^+ of one of the graphs Λ_i^+, for $i \leq n - 1$, whose image by $\tilde{h}$ lies in the stabiliser of v.

If v lies in T_2 itself, Lemma 7.11 gives that the intersection $S \cap \tilde{h}(A_1 * \ldots * A_n)$ is contained in the stabiliser of T_2, namely $\tilde{h}(A_n)$. Then, by Lemma 7.4, there is a conjugate S^+ of a surface type vertex group of Λ_n^+ whose image by $\tilde{h}$ lies in the stabiliser of v.

Finally, if v lies in a translate of T_1 or T_2 by an element α of $\tilde{h}(A_1 * \ldots * A_n)$, we apply the results above to the vertex $\alpha^{-1} \cdot v$. This is enough to prove the result. □

8 From preretractions to hyperbolic floors

In this section, we prove Proposition 5.11 and Proposition 5.12. From the existence of a non-injective preretraction $f : A \to A$, Proposition 5.11 deduces the existence of a retraction r which makes $(A, r(A), r)$ a hyperbolic floor, and from the existence of a non-injective preretraction $f : A \to G$, Proposition 5.12 deduces the existence of a non-injective preretraction p from A to a retract of G. In both proofs, the idea is to modify f into the map we want, namely the retraction r and the preretraction p respectively.

In the first subsection, we give a proof of Proposition 5.11 in a very simple case, where in particular the preretraction f is non-pinching. This example, as well as Section 7, illustrate how useful working with non-pinching maps can be. The aim of the second subsection will thus be to show how we can factor a preretraction $f : A \to G$ as $f' \circ \rho$, where f' is non-pinching with respect to some free factors of $\rho(A)$. This will be done by letting ρ kill elements corresponding to simple closed curves which lie in the kernel of f.

Recall that by definition of a hyperbolic floor, the image of the surface type vertex groups by the retraction should be non-abelian. In the third subsection, we worry about this, and give a criterion which will enable us later to guarantee that despite all the transformations we will make f undergo, non-abelianity of the images of surface groups is satisfied.

In the fourth subsection, we define a complexity on the set of non-injective preretractions $A \to A$. Note that with the hypotheses of Proposition 5.11, this set is non-empty since it contains f, so a maximal complexity non-injective preretraction exists: we give some properties of such a map. We then proceed in the fifth subsection to see how we can build from this maximal element a retraction $A \to A'$ which makes (A, A', r) a hyperbolic floor, thus proving Proposition 5.11.

The sixth subsection finally gives a proof of Proposition 5.12, it is independent of the fourth and fifth subsections.

8.1 A special case

We will prove 5.11 in the particular case where the JSJ-like decomposition Λ of A consists of two vertices, one of them of surface type, joined by a single edge. Denote by S the surface type vertex group, by Σ the corresponding surface, by R the non surface type vertex group, and by Z the edge group.

We assume moreover that the non-injective preretraction $f : A \to A$ we are given is non-pinching with respect to Λ (recall Definition 7.3). We will see in this case that up to conjugation of f, the triple (A, R, f) itself is a hyperbolic floor.

Note that up to conjugating f, we may assume that f restricts to the identity on R, so in particular $f(Z) = Z$. The surface group S acts via the map f on the tree T_Λ corresponding to Λ.

Lemma 8.1: *If S is elliptic in its action on T_Λ via f, then (A, R, f) is a hyperbolic floor.*

Proof. The image of S by f contains $f(Z)$, and $f(Z) = Z$ fixes the vertex v_R of T_Λ whose stabiliser is R. By 1-acylindricity next to surface type vertices, S fixes either v_R, or a translate by an element of R of the vertex v_S whose stabiliser is S.

If S fixes v_R, we have $f(S) \leq R$, but then $f(A) \leq R$ and f is the identity on R: the map f itself is a retraction $A \to R$, and Λ gives a hyperbolic floor decomposition .

Suppose now that S fixes v_S: we have $f(S) \leq S$. Now $f|_S$ is non-pinching, and sends boundary elements of S to boundary elements of S: by Lemma 3.10, $f(S)$ has finite index in S. If we see $f|_S$ as a map from S to itself, we have equality of the complexities, so by Lemma 3.12, $f|_S$ is bijective. Thus $f|_S$ is an isomorphism of surface group. However, it is easy to see now that f itself must be an

isomorphism, which contradicts its non-injectivity. If S fixes a translate of v_S by an element of R, we can apply a similar argument. □

So in the case where S is elliptic, we have indeed a hyperbolic floor. But in fact, we can show

Lemma 8.2: *The group S is elliptic in its action on T_Λ via f.*

Proof. Since f is the identity on the edge group Z, boundary elements of S are elliptic in this action. We may thus apply Lemma 3.5 to deduce the existence of an essential set of curves $\mathcal{C}$ on Σ and of a locally minimal equivariant map $t : T_\mathcal{C} \to T_\Lambda$ between the S-tree dual to $\mathcal{C}$ and T_Λ. The closures of connected components of the complement of $\mathcal{C}$ in Σ are subsurfaces of Σ whose fundamental group is elliptic in the action on T_Λ via f.

Suppose the fundamental group S_0 of one of these subsurface Σ_0 fixes the vertex v_S. This means that we have $f(S_0) \leq S$. Now boundary elements of S_0 correspond to boundary elements of S or to curves of $\mathcal{C}$ on Σ, in any case, they are sent to edge groups of Λ, that is to boundary elements of S. Since $f|_{S_0}$ is non-pinching, $f(S_0)$ has finite index in S by Lemma 3.10, so by Lemma 3.12, the complexity of Σ_0 is greater than that of Σ. This is not possible if Σ_0 is a proper subsurface of Σ: we conclude that $\mathcal{C}$ is empty, so that S is elliptic in its action on T_Λ via f. The case where S_0 fixes a translate of v_S can be dealt with similarly.

Suppose now that the fundamental groups of the subsurfaces whose interior are connected components of $\Sigma - \mathcal{C}$ all stabilise translates of v_R. By 1-acylindricity next to surface type vertices, it is straightforward to see that they must all stabilise v_R itself: again, this means $\mathcal{C}$ is empty. □

The general case is an elaboration on this example. First, we must deal with the fact that f might not be non-pinching: this is the object of the next subsection.

8.2 Pinching a set of curves

Let A be the fundamental group of a graph of groups with surfaces Λ which has infinite cyclic edge groups. Let $\mathcal{C}$ be a union of essential sets of curves, one on each of the surfaces of Λ.

We want to generalise Definition 5.24: we let $N(\mathcal{C})$ be the subgroup of A normally generated by the elements corresponding to the curves of $\mathcal{C}$, and we give

Definition 8.3: (pinching map) *We denote by $\rho_\mathcal{C}$ the quotient map $A \to A/N(\mathcal{C})$, and we call it the pinching map of A by $\mathcal{C}$.*

Denote by $\rho_\mathcal{C}(\Lambda)$ the quotient decomposition, namely the decomposition obtained from Λ by replacing each vertex group by its image by $\rho_\mathcal{C}$ (note that $\rho_\mathcal{C}$ is injective on edge groups of Λ).

Refine the graph of groups $\rho_\mathcal{C}(\Lambda)$ by replacing each vertex corresponding to a surface Σ of Λ by the graph of groups $\Gamma(\Sigma, \mathcal{C})$ defined in 5.25 (see Figure 4).

Definition 8.4: (pinching of a graph of groups with surfaces) *We call the graph of groups $\Lambda_\mathcal{C}$ thus obtained the pinching of Λ by $\mathcal{C}$.*

Let us see that this graph of groups decomposition gives us a decomposition of $\rho_\mathcal{C}(A)$ as a free product. Remove from $\Lambda_\mathcal{C}$ the interior of all the edges of the graphs $\Gamma(\Sigma, \mathcal{C})$ as well as the interior vertices: denote by $\Lambda_1, \ldots, \Lambda_l$ the various connected components. They are sub-graphs of groups of $\Lambda_\mathcal{C}$. We give them a structure of graph of groups with surfaces, by choosing as surface type vertices the vertices which belong to one of the subgraphs $\Gamma(\Sigma, \mathcal{C})$, and whose corresponding surface is not a cylinder or a Möbius band. Call $A_1, \ldots, A_l$ their fundamental groups.

The following lemma is a generalisation of Remark 5.26, its proof is straightforward.

Lemma 8.5: *If $\mathcal{C}$ is not empty, the complexity of the set containing all the surfaces of the graphs of groups Λ_i is strictly smaller than the complexity of the set of surfaces of Λ.*

Remark that if we collapse the edges of the subgraphs Λ_i in Λ, the graph of groups we get has trivial edges stabilisers, since the graphs $\Gamma(\Sigma, \mathcal{C})$ do. Picking a maximal subtree in it, and choosing a

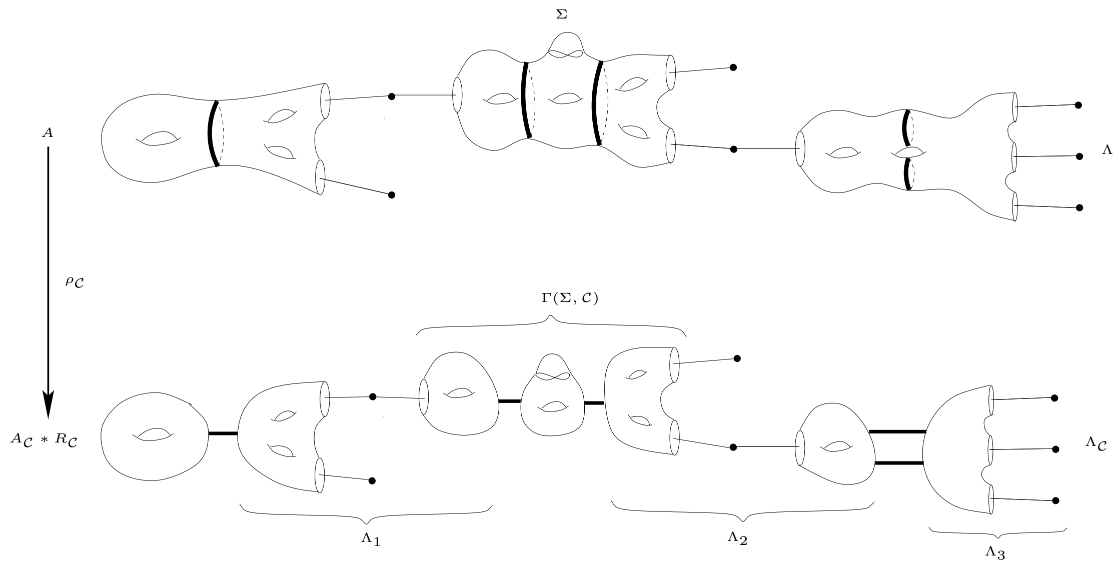

Figure 4: The pinching of Λ by $\mathcal{C}$.

lift in the $\rho_{\mathcal{C}}(A)$-tree corresponding to this graph of group gives us an identification of the groups A_i to subgroups of $\rho_{\mathcal{C}}(A)$, and a free product decomposition of $\rho_{\mathcal{C}}(A)$ of the form

$$\rho_{\mathcal{C}}(A) = (A_1 * \ldots * A_l) * (S_1 * \ldots * S_p) * (Z_1 * \ldots * Z_q) \qquad (\dagger)$$

where the groups S_j are fundamental groups of closed surfaces which are not spheres, corresponding to interior vertices of the graphs of groups $\Gamma(\Sigma, \mathcal{C})$, and each group Z_k is the infinite cyclic subgroup of $\rho_{\mathcal{C}}(A)$ corresponding to an edge lying outside the maximal subtree.

Definition 8.6: (pinching decomposition of $\rho_{\mathcal{C}}(A)$) *We call the free product decomposition* ($\dagger$) *a pinching decomposition of $\rho_{\mathcal{C}}(A)$ with respect to $\mathcal{C}$.*

Note that different choices of maximal subtree and different lifts in the $\rho_{\mathcal{C}}(A)$-tree give us different pinching decompositions of $\rho_C(A)$. Finally, we will use the following notations

$$\begin{aligned} A_{\mathcal{C}} &:= A_1 * \ldots * A_l \\ R_{\mathcal{C}} &:= (S_1 * \ldots * S_p) * (Z_1 * \ldots * Z_q). \end{aligned}$$

Some of the vertex groups of the graphs of groups Λ_i which are almost isomorphic images of surface group of Λ by the map $\rho_{\mathcal{C}}$ will play a particular role in the third section. We thus define

Definition 8.7: (intact vertex of Λ_i) *Let S_0 be a vertex group of Λ_i which belongs to one of the subgraphs $\Gamma(\Sigma, \mathcal{C})$. Suppose that the graph of groups $\Gamma(\Sigma, \mathcal{C})$ is a tree of groups, all of whose vertex groups except S_0 are trivial or isomorphic to $\mathbb{Z}/2\mathbb{Z}$. We call the vertex of Λ_i corresponding to S_0 an intact vertex.*

If S_0 is non-cyclic, it is conjugate to a surface type vertex group of one of the graphs of groups Λ_i. In this case we call the corresponding vertex of Λ_i an intact surface type vertex of Λ_i, and we call the corresponding surface an intact surface of Λ_i.

We said that our strategy was to factor the non-injective preretraction f as $f = f' \circ \rho$, where the map f' is non-pinching with respect to a suitable graph of groups. The map ρ should thus be the quotient of A by a maximal set of elements coming from simple closed curves pinched by f. This is a generalisation of Definition 5.23:

Definition 8.8: (maximal essential set of curves pinched by f) *Let A be the fundamental group of a graph of groups with surfaces Λ, whose edge groups are infinite cyclic. Let $f: A \to G$ be a map which is injective on edge groups.*

A (maximal) essential set of curves pinched by f is the union over all the surfaces $\{\Sigma_j\}_{1\leq j\leq r}$ of Λ of (maximal) essential sets of curves pinched by $f|_{\pi_1(\Sigma_j)}$ on Σ_j.

Remark 8.9: *In the setting of Definition 8.8, if $\mathcal{C}$ is a maximal essential set of curves pinched by f, the map f factors as $f' \circ \rho_{\mathcal{C}}$. Now if we consider a pinching decomposition of the form (†) of A with respect to $\mathcal{C}$, the maps $f'|_{A_i}$ are non-pinching with respect to the graphs of groups Λ_i.*

We saw that working with non-pinching maps was easier, thus to prove Proposition 5.11, it seems like a good idea to take immediately a pinching map for an essential set of curves pinched by the preretraction f, and to restrict our attention to the map $f'|_{A_{\mathcal{C}}}$. However, we need to bear in mind that the maps we produce should send surface type vertex groups to non-abelian images. For this, we 'set aside' $R_{\mathcal{C}}$, and once we will have modified $f'|_{A_{\mathcal{C}}}$ as we need, we will be able to use this factor we set aside to guarantee non-abelianity of the image of surface type vertex groups. This trick is the object of the next section.

8.3 Non-abelianity of surfaces

We now give a criterion which will prove very useful in the proofs of Propositions 5.11 and 5.12. It implies that if we have a map g from $A_{\mathcal{C}}$ to a torsion-free hyperbolic group G, as long as intact vertex groups are not sent to abelian images, we can extend g to a map from $\rho_{\mathcal{C}}(A) = A_{\mathcal{C}} * R_{\mathcal{C}}$ to G whose composition with $\rho_{\mathcal{C}}$ sends all the surface type vertex groups of Λ to non-abelian images.

Lemma 8.10: *Let A be a group which admits a JSJ-like decomposition Λ. Let $\mathcal{C}$ be an essential set of curves on the surfaces of Λ. Choose a pinching decomposition (†) of $\rho_{\mathcal{C}}(A)$. Suppose g is a map from $A_{\mathcal{C}} = A_1 * \ldots * A_l$ to a torsion-free hyperbolic group G such that*

- *g is injective on edge groups of the graphs Λ_i;*
- *if two edge groups of some of the graphs Λ_i have disjoint conjugacy classes in $A_{\mathcal{C}}$, their images by g have disjoint conjugacy classes in G;*
- *the images by g of intact vertex groups are non-abelian.*

Then for any subset G_0 of G which is not contained in a finite union of cyclic subgroup, there is a map $\tau : (S_1 * \ldots * S_p) * (Z_1 * \ldots * Z_q) \to G_0$ such that the map $(g * \tau) \circ \rho_{\mathcal{C}} : A \to G$ sends surface type vertex groups of Λ on non-abelian images.

$$\begin{tikzcd} & A_{\mathcal{C}} \arrow[r, "g"] & G \\ A \arrow[r, "\rho_{\mathcal{C}}"] & * & \\ & R_{\mathcal{C}} \arrow[uur, dotted, "\exists\tau"'] & \end{tikzcd}$$

Proof. Let Σ be a surface of Λ, denote by S the corresponding surface type vertex group. We will show that, provided the image of the factors S_i and Z_j by τ is not contained in a given finite union of maximal cyclic subgroups, then $(g * \tau) \circ \rho_{\mathcal{C}}(S)$ is not abelian.

If $\Gamma(\Sigma, \mathcal{C})$ is a tree of groups, all of whose vertex groups except one are fundamental groups of spheres and projective planes, then $\rho_{\mathcal{C}}(S)$ contains an intact vertex group Q. The image of Q by g is non-abelian, so the image of S by $(g * \tau) \circ \rho_{\mathcal{C}}$ is non-abelian regardless of the choice of τ. We may now assume that $\Gamma(\Sigma, \mathcal{C})$ is not a tree of groups all of whose vertex groups except one are trivial or $\mathbb{Z}/2\mathbb{Z}$.

Suppose now that $\Gamma(\Sigma, \mathcal{C})$ is not a tree of groups, or that it contains an interior vertex whose group is the fundamental group of a closed surface of positive genus. Let z be the generator of an edge group adjacent to $\Gamma(\Sigma, \mathcal{C})$ in $\Lambda_{\mathcal{C}}$. Up to replacing S by a conjugate, we see that $\rho_{\mathcal{C}}(S)$ contains both z, and either one of the factors Z_j, or one of the factors S_j of the pinching decomposition. If the image by

τ of this factor lies outside of the maximal cyclic subgroup $C_{g(z)}$ containing $g(z)$, the image of S by $(g * \tau) \circ \rho_{\mathcal{C}}$ is not abelian.

We may thus assume $\Gamma(\Sigma, \mathcal{C})$ is a tree of groups, that all its interior vertices are finite, and that it has at least two non-interior vertices v_1 and v_2 with corresponding groups S_1 and S_2. Let B_i be a maximal boundary subgroup of S_i for $i = 1, 2$. Note that each B_i is an edge group of Λ_{j_i} for some index j_i. If B_1 and B_2 have disjoint conjugacy classes in $A_{\mathcal{C}}$, their images by g have disjoint conjugacy classes in G by hypothesis. Thus $g(\rho_{\mathcal{C}}(S))$ is not abelian. If some element of B_1 can be conjugated into B_2 by an element of $A_{\mathcal{C}}$, we must have $\Lambda_{j_1} = \Lambda_{j_2}$ since $A_{\mathcal{C}}$ is the free product of the groups A_i. Then the edges corresponding to B_1 and B_2 must be adjacent to a same Z type vertex v of Λ_{j_1}. Now there is a path in $\Gamma(\Sigma, \mathcal{C})$ between v_1 and v_2, so there is a loop in $\Lambda_{\mathcal{C}}$ based at v, we may assume this loop contains exactly one edge which does not lie in the maximal subtree we chose for our pinching decomposition. Finally, we see that (up to conjugation), $\rho_{\mathcal{C}}(S)$ contains both an element z of the group corresponding to v, and its conjugate tzt^{-1} by the generator t of one of the factor Z_i of the pinching decomposition. If $\tau(t)$ does not lie in the maximal cyclic subgroup $C_{g(z)}$ containing $g(z)$, then $(g*\tau)(z)$ and $(g * \tau)(tzt^{-1})$ do not commute, so $(g * \tau) \circ \rho_{\mathcal{C}}(S)$ is not abelian. ☐

A straightforward growth argument gives

Lemma 8.11: *If G is a torsion-free hyperbolic group, a non-cyclic subgroup G' of G is not contained in a finite union of cyclic groups.*

Let us now show a lemma which implies in particular that if $f : A \to G$ is a preretraction which factors as $f' \circ \rho_{\mathcal{C}}$ where $\mathcal{C}$ is a maximal essential set of curves pinched by f, and if G is torsion-free hyperbolic, then intact vertex groups have non-abelian images by f', so $f'|_{A_{\mathcal{C}}}$ satisfies one of the conditions of Lemma 8.10.

Lemma 8.12: *Let Λ be a JSJ-like decomposition of a group A. Let G be a torsion-free hyperbolic group. Suppose $f : A \to G$ is a morphism which sends surface type vertex groups of Λ onto non-abelian images, and is injective on edge groups. Let $\mathcal{C}$ be a maximal essential set of curves pinched by f, so that f factors as $f' \circ \rho_{\mathcal{C}}$.*

If S is an intact vertex group of Λ_i, then it is of surface type. Moreover, if Σ is the surface corresponding to S, and if $\Delta(\Sigma, \mathcal{C}^+)$ is a graph of group decomposition dual to some essential set of curves $\mathcal{C}^+$ on Σ, there is at least one vertex group of $\Delta(\Sigma, \mathcal{C}^+)$ whose image by f' is non-abelian.

Proof. Let S be an intact vertex group of Λ_i. We show first that $f'(S)$ is non-abelian (this will imply in particular that S itself is non-cyclic). The group S is the unique infinite vertex group of one of the graph of groups of the form $\Gamma(\Sigma_0, \mathcal{C})$ for some surface Σ_0 of Λ, and we know that the graph underlying $\Gamma(\Sigma_0, \mathcal{C})$ is a tree of groups. Since G is torsion-free, the image by f' of the other finite vertex groups of $\Gamma(\Sigma_0, \mathcal{C})$ are trivial, so that the image of the fundamental group $\rho_{\mathcal{C}}(S_0)$ of $\Gamma(\Sigma, \mathcal{C})$ by f' is exactly the image of S by f': we have $f(S_0) = f'(\rho_{\mathcal{C}}(S_0)) = f'(S)$. Now since S_0 is a surface type vertex group of Λ, its image by f is non-abelian, which proves the claim.

Suppose now that the image by f' of each one of the vertex groups of $\Delta(\Sigma, \mathcal{C}^+)$ is abelian. Since G is hyperbolic, these images are in fact infinite cyclic. Since f' is non-pinching with respect to Λ_i, and since G is torsion-free, the edge groups of $\Delta(\Sigma, \mathcal{C}^+)$ are sent injectively into G by f'. This gives a graph of group decomposition of $f'(S)$ all of whose vertex and edge groups are infinite cyclic, so $f'(S)$ is a generalised Baumslag-Solitar group. In a generalised Baumslag-Solitar group, the commensurator of an elliptic element is the whole group (see for example [For02]). But in a torsion-free hyperbolic group, commensurators of elements are cyclic groups. This contradicts the non-abelianity of $f'(S)$, thus at least one of the vertex groups of $\Delta(\Sigma, \mathcal{C}^+)$ has non-abelian image by f'. ☐

Finally, we show that if A is a retract of G, and f a preretraction $A \to G$, then the other hypothesis of Lemma 8.10 is satisfied, namely f' preserves disjointness of conjugacy classes of edge groups of the graphs Λ_i.

Lemma 8.13: *Let A be a retract of a torsion-free hyperbolic group G. Let Λ be a JSJ-like decomposition of A. Suppose $f : A \to G$ is a preretraction. Let $\mathcal{C}$ be a maximal essential set of curves pinched by f, so that f factors as $f' \circ \rho_{\mathcal{C}}$.*

Then if two edge groups of some of the graphs Λ_i have disjoint conjugacy classes in $A_{\mathcal{C}}$, their images by f' have disjoint conjugacy classes in G.

Proof. Let B_1 and B_2 be edge groups of Λ_{j_1} and Λ_{j_2}. There exists edges e_1 and e_2 of T_Λ such that $B_i = \rho_{\mathcal{C}}(\mathrm{Stab}(e_i))$ for $i = 1, 2$. If B_1 and B_2 have disjoint conjugacy classes in $\rho_{\mathcal{C}}(A)$, then $\mathrm{Stab}(e_1)$ and $\mathrm{Stab}(e_2)$ have disjoint conjugacy classes in A. Now f sends edge groups to conjugate of themselves, so $f'(B_i) = f(\mathrm{Stab}(e_i))$ is a conjugate of $\mathrm{Stab}(e_i)$ for $i = 1, 2$. But A is a retract of G, thus the fact that $\mathrm{Stab}(e_1)$ and $\mathrm{Stab}(e_2)$ have disjoint conjugacy classes in A implies that they have disjoint conjugacy classes also in G. $\square$

8.4 Maximal preretractions

Until the end of this subsection, we let A be a torsion-free hyperbolic group with a JSJ-like decomposition Λ, and we assume that there exists at least one non-injective preretraction $A \to A$ with respect to Λ. Note that Λ must have at least one surface type vertex: if not, by Proposition 6.1, any preretraction $A \to A$ is in fact an isomorphism.

Definition 8.14: (the set $L(f)$) *If $f : A \to A$ is a preretraction with respect to Λ, we denote by $L(f)$ the set of surfaces of Λ such that for at least one of the corresponding vertex groups S, the intersection $f(A) \cap S$ is not contained in a boundary subgroup of S.*

Consider the set of tuples $(f, \mathcal{C}, \mathcal{C}^+)$ for which

- f is a non-injective preretraction $A \to A$;
- $\mathcal{C}$ is a maximal essential set of curves pinched by f, so that there exists $f' : \rho_{\mathcal{C}}(A) \to A$ with $f = f' \circ \rho_{\mathcal{C}}$;
- $\mathcal{C}^+$ is an essential set of curves on the surfaces of the graph of groups Λ_i obtained in the pinching of Λ by $\mathcal{C}$, such that $\mathcal{C}^+$ gives elliptic refinements Λ_i^+ of each Λ_i relatively to f' and Λ.

We say that such a tuple $(f, \mathcal{C}, \mathcal{C}^+)$ is greater than another element $(g, \mathcal{D}, \mathcal{D}^+)$ if $\mathcal{C}$ strictly contains $\mathcal{D}$, or if they are equal and $\mathcal{C}^+$ strictly contains $\mathcal{D}^+$, or if they too are equal, and $L(f)$ is contained in $L(g)$ (note the inversion).

A preretraction f for which there exists $\mathcal{C}$ and $\mathcal{C}^+$ such that $(f, \mathcal{C}, \mathcal{C}^+)$ is a maximal element for this ordering is called a maximal non-injective preretraction. Such an element must exist, indeed, we assumed the set of non-injective preretractions to be non-empty, and the cardinal of an essential set of curves on a finite set of surfaces is bounded.

For the rest of this subsection, we let $f : A \to A$ be a maximal non-injective preretraction for the sets of curves $\mathcal{C}$ and $\mathcal{C}^+$. Build the pinching $\Lambda_{\mathcal{C}}$ of Λ by $\mathcal{C}$ (as given by Definition 8.4), a pinching decomposition of $\rho_{\mathcal{C}}(A)$ (as given by Definition 8.6), and elliptic refinements Λ_i^+ of the graphs of groups Λ_i given by $\mathcal{C}^+$. By Remark 7.2, we have locally minimal equivariant maps $t_i^+ : T_{\Lambda_i^+} \to T_\Lambda$.

A very important property of such a maximal element is given by

Lemma 8.15: *For any surface Σ of Λ, the following are equivalent:*

(i) $\Sigma \in L(f)$;

(ii) there is a surface type vertex v of T_Λ corresponding to Σ which lies in the image of one of the maps $t_i^+ : T_{\Lambda_i^+} \to T_\Lambda$;

(iii) there is a surface type vertex group S corresponding to Σ, and a surface type vertex group S^+ of one of the elliptic refinements Λ_i^+ such that $f'(S^+)$ is a subgroup of finite index of S.

Proof. The equivalence between (ii) and (iii) is given by Lemma 7.4. It is clear that (iii) implies (i). Let us show that the converse is true.

If $m = 1$, there is only one component Λ_1, the result is given by Lemma 7.4. More generally, the idea is to use Lemma 7.9 to build from f a map F which satisfies this, and which turns out to be also a maximal preretraction. The maximality of f will then imply that f itself satisfies the lemma.

If $m \geq 2$, the set of curves $\mathcal{C}$ is not empty, so in particular by Lemma 8.5, the surface complexity of each of the graph of groups Λ_i is smaller than the surface complexity of Λ. Consider the map $h = f'|_{A_\mathcal{C}} : A_1 * \ldots * A_m \to A$. The hypotheses of Lemma 7.9 are satisfied, so we can find a map $\tilde{h} : A_1 * \ldots * A_m \to A$ such that $\tilde{h}|_{A_i}$ coincides with $f'|_{A_i}$ up to conjugation, and if S is a surface type vertex group of Λ whose intersection with $\tilde{h}(A_1 * \ldots * A_l)$ is not contained in a boundary subgroup of S, then there is a surface type vertex group S^+ of one of the graphs of groups Λ_i^+ whose image by $\tilde{h}$ has finite index in a conjugate of S. We also know that $\tilde{h}(A_1 * \ldots * A_m)$ is the free product of the $\tilde{h}(A_i)$, so in particular it is not abelian since we assumed $m \geq 2$.

By Lemma 8.12, the map f' sends intact vertex groups of the graph of groups Λ_i to non-abelian images. The image by $\tilde{h}$ of an intact vertex group of one of the graphs Λ_i is just a conjugate of its image by f', so it is also non-abelian. By Lemma 8.13, f' preserves the disjointness of conjugacy classes of edge groups of Λ_i: again this clearly implies $\tilde{h}$ itself satisfies this hypothesis. Thus the hypotheses of Lemma 8.10 are satisfied, and there exists a map $\tau : R_\mathcal{C} \to \tilde{h}(A_1 * \ldots * A_l)$ such that the map $F = (\tilde{h} * \tau) \circ \rho_\mathcal{C}$ sends surface type vertex groups of Λ to non-abelian images. We now want to see that $(F, \mathcal{C}, \mathcal{C}^+)$ is a maximal non-injective preretraction.

It is easy to check that F restricts to a conjugation on each non surface type vertex group of Λ, so that F is a preretraction. The map F factors through $\rho_\mathcal{C}$, so F is in fact a non-injective preretraction, and the curves of $\mathcal{C}$ are pinched by F. By maximality of f, we see that $\mathcal{C}$ is a maximal essential set of curves pinched by F. The map f' sends elements corresponding to curves of $\mathcal{C}^+$ to edge groups of Λ, thus so does the map $\tilde{h}$. Similarly, by maximality of f, the curves $\mathcal{C}^+$ must give elliptic refinements of the graph of groups Λ_i with respect to $\tilde{h}$ and Λ.

Now, let Σ be a surface of Λ which lies in $L(F)$: there is a surface type vertex group S of Λ corresponding to Σ whose intersection with $F(A) = \tilde{h}(A_1 * \ldots * A_l)$ is not contained in a boundary subgroup. Since we chose $\tilde{h}$ according to Lemma 7.9, there is a surface type vertex group S^+ of one of the graphs of groups Λ_i^+ such that $\tilde{h}(S^+)$ is a subgroup of finite index of S. But on A_i, the maps $\tilde{h}$ and f' coincide up to conjugation: thus $f'(S^+)$ is a subgroup of finite index of some conjugate of S. We have shown that to any surface Σ which lies in $L(F)$ corresponds a group S which admits as a subgroup of finite index the image by f' of a surface type vertex group S^+ of one of the graphs of groups Λ_i^+.

This implies first that $L(F) \subseteq L(f)$. By maximality of f, we see that this must in fact be an equality. But then if Σ is in $L(f)$, it is also in $L(F)$, so there is a group S with corresponding surface Σ which admits as a subgroup of finite index the image by f' of a surface type vertex group S^+ of one of the graphs of groups Λ_i^+: we see that (iii) must hold. □

From this we deduce in particular

Lemma 8.16: *The set $L(f)$ does not contain all the surfaces of Λ.*

Proof. Suppose that $L(f)$ contains all the surfaces of Λ. By Lemma 8.15, for every surface Σ of Λ, there exists a surface type vertex group S^+ of one of the graphs of groups Λ_i^+ such that $f(S^+)$ is a subgroup of finite index of one of the surface type vertex groups S corresponding to Σ. Moreover, f' sends boundary subgroups of S^+ to boundary subgroups of S. By Lemma 3.12, the complexity of the surface Σ^+ corresponding to S^+ is greater than or equal to that of Σ, and if we have equality, $f'|_{S^+}$ is an isomorphism onto S. This implies that the complexity of the set of all the surfaces of the Λ_i^+ is greater than the surface complexity of Λ. By Lemma 7.7 and Lemma 8.5, we see that we must in fact have equality. This implies that the sets $\mathcal{C}$ and $\mathcal{C}^+$ are empty, and that each surface type vertex group of Λ is sent isomorphically onto a surface type vertex group of Λ by f, non-conjugate surface type vertex groups being sent to non-conjugate surface type vertex group.

Thus some power of f sends each surface type vertex group of Λ isomorphically on a conjugate of itself, and restricts to conjugation on each non surface type vertex group. By Proposition 6.1, it is an isomorphism. This contradicts the non-injectivity of f. $\square$

We now want to define applications $P(f,k) : A \to A$ which we call pseudo-powers of f. Indeed, we need to iterate f, but we want the result to still be a preretraction, this is why we cannot take simply the powers of f since they might send surface type vertex groups onto abelian images.

We define $P(f,k)$ by induction as follows. Let $P(f,1) = f$. If $P(f,k-1)$ is defined, and is a maximal preretraction $A \to A$ we consider the map $P(f,k-1) \circ (f'|_{A_{\mathcal{C}}}) : A_{\mathcal{C}} \to A$.

Lemma 8.17: *The map $P(f,k-1) \circ (f'|_{A_{\mathcal{C}}})$ sends intact vertex groups of the graphs of groups Λ_i to non-abelian images, and preserves disjointness of conjugacy classes of edge groups of the graphs of groups Λ_i.*

Proof. If S is an intact vertex of one of the graphs of groups Λ_i, we know by Lemma 8.12 that it is in fact a surface type vertex. If Σ denotes the surface corresponding to S, the group S inherits a decomposition $\Delta(\Sigma, \mathcal{C}^+)$ from the elliptic refinement Λ_i^+. Again by Lemma 8.12, we know that there is at least one of the vertex groups S_0 of $\Delta(\Sigma, \mathcal{C}^+)$ whose image by f' is non-abelian. If $f'(S_0)$ lies in a non-surface type vertex group of Λ, the preretraction $P(f,k-1)$ is injective on $f'(S_0)$, so $P(f,k-1) \circ f'(S)$ is non-abelian. If $f'(S_0)$ lies in a surface type vertex group S_1 of Λ, it must be with finite index by Lemma 3.10 since f' is non-pinching on Σ, and sends edge groups of $\Delta(\Sigma, \mathcal{C}^+)$ and boundary elements of Σ on edge groups of Λ. Now $P(f,k-1) \circ f'(S_0)$ is a subgroup of finite index of $P(f,k-1)(S_1)$, which is non-abelian since $P(f,k-1)$ is a preretraction. Since we are in a torsion-free hyperbolic group, $P(f,k-1) \circ f'(S)$ itself is non-abelian.

Note that f' sends edge groups of the graphs of groups Λ_i to edge groups of Λ, moreover by Lemma 8.13 it preserves disjointness of their conjugacy classes. Since $P(f,k-1)$ sends edge groups of Λ to conjugates of themselves, $P(f,k-1) \circ f'$ preserves disjointness of conjugacy classes of edge groups of the graphs Λ_i. $\square$

We will now build $P(f,k)$. Since Λ admits at least one surface type vertex, and $P(f,k-1)$ is a preretraction with respect to Λ, we see that $P(f,k-1)(A)$ is not abelian. We can thus apply Lemma 8.10 to $P(f,k-1) \circ f'|_{A_{\mathcal{C}}}$, this tells us we can find a map $\tau : R_{\mathcal{C}} \to P(f,k-1)(A)$ such that the map

$$P(f,k) = [(P(f,k-1) \circ f'|_{A_{\mathcal{C}}}) * \tau] \circ \rho_{\mathcal{C}}$$

sends surface type vertex groups on non-abelian images.

$$\begin{tikzcd}
 & A_{\mathcal{C}} \arrow[r, "f'"] & f'(A_{\mathcal{C}}) \arrow[r, "P(f,k-1)"] & A' \\
A \arrow[r, "\rho_{\mathcal{C}}"] & * & & \\
 & R_{\mathcal{C}} \arrow[rruu, dotted, "\exists\tau"'] & &
\end{tikzcd}$$

Let us now see that the morphism $P(f,k)$ thus defined is a maximal non-injective preretraction. It is easy to see that $P(f,k)$ sends non surface type vertex groups on conjugates of themselves, so it is a preretraction. If $\mathcal{C}$ is empty, $f' = f$ so $P(f,k)$ is not injective since f is not injective.

Since $P(f,k)$ factors through $\rho_{\mathcal{C}}$, it pinches the curves in $\mathcal{C}$, so by maximality of $P(f,k-1)$ the set $\mathcal{C}$ is a maximal essential set of curves pinched by $P(f,k)$. Similarly since $P(f,k-1)$ conjugates edge groups of Λ, $P(f,k-1) \circ f'|_{A_{\mathcal{C}}}$ sends elements corresponding to curves of $\mathcal{C}^+$ to edge groups of Λ, so by maximality of $P(f,k-1)$, the set $\mathcal{C}^+$ is a maximal essential set of curves that give an elliptic refinement of the graphs of groups Λ_i with respect to $P(f,k-1) \circ f'|_{A_{\mathcal{C}}}$ and Λ. Finally, the image of $P(f,k)$ is contained in the image of $P(f,k-1)$, so $L(P(f,k)) \subseteq L(P(f,k-1))$, and by maximality of $P(f,k-1)$ this is in fact an equality. Thus $P(f,k)$ is a maximal non-injective preretraction.

Using pseudo-powers, we can now show

Lemma 8.18: *If f is a maximal non-injective preretraction, it sends surface type vertex groups corresponding to distinct surfaces of $L(f)$ isomorphically onto surface type vertex groups corresponding to distinct surfaces of $L(f)$.*

Proof. We have just seen that $P(f,2)$ is also a maximal preretraction, for the same sets $\mathcal{C}$ and $\mathcal{C}^+$. Thus $P(f,2)$ factors through $\rho_{\mathcal{C}}$, we write $P(f,2) = [P(f,2)]' \circ \rho_{\mathcal{C}}$. Recall that $P(f,2) = [(f \circ f'|_{A_{\mathcal{C}}}) * \tau] \circ \rho_{\mathcal{C}}$ so that $[P(f,2)]'|_{A_{\mathcal{C}}} = f \circ f'|_{A_{\mathcal{C}}}$. Let Σ be a surface of $L(f)$.

Since $L(f) = L(P(f,2))$, and using Lemma 8.15, we see that there is a group S corresponding to the surface Σ, and a surface type vertex group S^+ of Λ_i^+ for some i, such that $f \circ f'(S^+)$ is a subgroup of finite index of S. Consider now $f'(S^+)$: it is elliptic in Λ since the Λ_i^+ are elliptic refinements relative to both f' and $[P(f,2)]'$.

It cannot lie in a non surface type vertex group of Λ, since these are sent to conjugates of themselves by f, and $f(f'(S^+))$ lies in S. Thus it lies in a surface type vertex group S_1 of Λ, and by Lemma 3.10, it is a subgroup of finite index of S_1. This implies in particular that the surface corresponding to S_1 is in $L(f)$.

Now $f(S_1)$ contains a subgroup of finite index, namely $f(f'(S^+))$, which is elliptic in T_Λ: thus $f(S_1)$ itself is elliptic. Thus it lies in a vertex stabiliser of T_Λ, which must be S. Since $f(f'(S^+))$ has finite index in S, so does $f(S_1)$. By Lemma 3.12, the complexity of the surface Σ_1 corresponding to S_1 is at least that of the surface Σ corresponding to S, and if we have equality, $f|_{S_1}$ is an isomorphism onto S.

Thus to each surface Σ in $L(f)$ corresponds a surface Σ_1 in $L(f)$ whose complexity is greater, and such that any group S_1 corresponding to Σ_1 has image by f lying in a group S corresponding to Σ. In particular, the map $\Sigma \mapsto \Sigma_1$ is injective. Since it is a map $L(f) \to L(f)$, it is a bijection, thus for any surface Σ of $L(f)$, we must have $k(\Sigma) = k(\Sigma_1)$. This implies that f sends each surface type vertex group whose surface is in $L(f)$ isomorphically onto a surface type vertex group whose surface is in $L(f)$, and any two such groups which are non-conjugate are sent to non-conjugate images. □

We get

Corollary 8.19: *Let A be a torsion-free hyperbolic group which admits a cyclic JSJ-like decomposition Λ. If there exists a non-injective preretraction $A \to A$, then there exists a maximal non-injective preretraction f which sends each surface type vertex group whose corresponding surface is in $L(f)$ isomorphically onto a conjugate of itself.*

Proof. Let g be a maximal non-injective preretraction: by Lemma 8.18, it sends sends surface type vertex groups corresponding to distinct surfaces of $L(g)$ isomorphically onto surface type vertex groups corresponding to distinct surface of $L(g)$. There exists an integer k such that the pseudo-power $f = P(g,k)$ sends surface type vertex groups whose corresponding surface is in $L(g)$ isomorphically onto conjugates of themselves. By maximality of g, we have $L(P(g,k)) = L(g)$. □

8.5 Proof of Proposition 5.11

Let A be a torsion-free hyperbolic group which admits a cyclic JSJ-like decomposition Λ, and assume that there exists at least one non-injective preretraction $A \to A$ with respect to Λ. By Corollary 8.19, there exists a non-injective maximal preretraction $f : A \to A$ which sends every group corresponding to a surface of $L(f)$ isomorphically to a conjugate of itself.

If we remove from Λ the surface type vertices corresponding to surfaces which do *not* lie in $L(f)$, as well as the open edges adjacent to these vertices, we get a subgraph of Λ, whose connected components we denote by $\Gamma_1, \ldots, \Gamma_m$. They are sub-graphs of groups of Λ, we denote their fundamental groups by $H_1, \ldots, H_m$. The surfaces of $\Gamma_1, \ldots, \Gamma_m$ are exactly the surfaces of $L(f)$, so f is injective on the corresponding surface groups, and in particular, no curves of $\mathcal{C}$ lie on the surfaces of $\Gamma_1, \ldots, \Gamma_m$.

Call Γ the graph of groups with surfaces obtained by collapsing in Λ all the edges of the subgraphs Γ_i. If we choose a maximal subtree in Γ, as well as a lift to the corresponding tree T_Γ, we identify the groups H_i to subgroups of A. Given a preferred non surface type vertex group R_0 of Λ, we can do this in such a way that R_0 lies in one of the subgroups H_i. The subgroup of A generated by $H_1, \ldots, H_m$ will be our retract A'.

Note that T_Γ is bipartite, in the sense that any edge has one end whose vertex group is a conjugate of one of the subgroups H_i, and one end which is a surface type vertex of Λ whose corresponding surface is not in $L(f)$.

Lemma 8.20: *The map f sends each H_i isomorphically onto a conjugate of itself.*

Proof. A vertex group of Γ_i is either a non-surface type vertex group, or a surface type vertex group whose corresponding surface is in $L(f)$: in both cases, it is sent by f isomorphically on a conjugate of itself in A. Let G_v and G_w be two adjacent vertex groups of Γ_i, with $f(G_v) = g_v G_v g_v^{-1}$ and $f(G_w) = g_w G_w g_w^{-1}$. By injectivity of f on edge groups, bipartism of Γ and 1-acylindricity near surface type vertices, we can see that $g_w^{-1} g_v$ is in H_i, so that $f(G_w)$ and $f(G_v)$ lie in the same conjugate of H_i. By connectedness of Γ_i, the images of all the vertex groups of Γ_i by f lie in the same conjugate of H_i.

Thus, $f|_{H_i}$ composed by the conjugation by g_v^{-1} restricts to a conjugation by an element of H_i on non-surface type vertex groups of Γ_i, and sends surface type vertex groups isomorphically on conjugates of themselves by an element of H_i.

As fundamental groups of subgraphs of groups of Λ, the groups H_i are quasiconvex in A, thus they are hyperbolic (see for example Proposition 4.2, Chapter 10 of [CDP90]). Note also that the decomposition Γ_i is a JSJ-like decomposition for H_i. We can now apply Proposition 6.1 to conclude that $f|_{H_i}$ composed with the conjugation by g_v^{-1} is an isomorphism $H_i \to H_i$. Thus f itself sends each H_i isomorphically onto a conjugate of itself. $\square$

We choose a pinching decomposition (recall Definition 8.6)

$$(A_1 * \ldots * A_l) * (S_1 * \ldots * S_p) * (Z_1 * \ldots * Z_q)$$

of $\rho_{\mathcal{C}}(A)$, and we let f' be such that $f = f' \circ \rho_{\mathcal{C}}$. Recall that the set $\mathcal{C}^+$ gives us elliptic refinements Λ_i^+ for each Λ_i with respect to f' and Λ.

Lemma 8.21: *For each j, the image of A_j by f' lies in a conjugate of one of the subgroups H_i.*

Proof. For each Λ_j^+, we have a locally minimal equivariant map $t_j^+ : T_{\Lambda_j^+} \to T_\Lambda$. If a surface type vertex of T_Λ lies in the image of t_j^+, by Lemma 8.15, $f(A_j)$ intersects its stabiliser with finite index, so the corresponding surface lies in $L(f)$. Thus the image of t_j^+ contains none of the vertices corresponding to surfaces which are not in $L(f)$. Since $t_j^+(T_{\Lambda_j^+})$ is connected, this implies that the image of A_j by f lies in a conjugate of one of the subgroups H_i. $\square$

Fix an index i. Recall that no curve of $\mathcal{C}$ lies on the surfaces of the graphs of groups $\Gamma_1, \ldots, \Gamma_m$, so that $\rho_{\mathcal{C}}(H_i)$ lies in a conjugate of one of the subgroups A_{j_i}. We just saw that H_i is sent isomorphically onto a conjugate of itself by f, and that $f'(A_{j_i})$ lies in a conjugate of one of the subgroups H_k: we must have that $f'(A_{j_i})$ lies in a conjugate of H_i. In particular, the application $i \mapsto j_i$ is injective. Conversely, each A_j contains a conjugate of one of the $\rho_{\mathcal{C}}(H_i)$. Up to renumbering, we may thus assume that $\rho_{\mathcal{C}}(H_i)$ is contained in a conjugate of A_i.

Lemma 8.22: *The subgroup A' of A generated by $H_1, \ldots, H_m$ is the free product $H_1 * \ldots * H_m$, and it is a proper subgroup of A.*

Proof. Recall that the group $A_{\mathcal{C}}$ generated by the groups A_i is in fact the free product of the groups A_i. Since the A_i form a free product in $\rho_{\mathcal{C}}(A)$, the group $\rho_{\mathcal{C}}(A')$ generated by the subgroups $\rho_{\mathcal{C}}(H_i)$ is in fact the free product of the subgroups $\rho_{\mathcal{C}}(H_i)$. Since $\rho_{\mathcal{C}}$ is injective on H_i, the H_i themselves form a free product.

Now the fact that the list $L(f)$ does not contain all the surfaces of Λ, implies that the group A' is a proper subgroup of A. $\square$

We now want to understand the image of $f'|_{A_{\mathcal{C}}}$. For each i, we have $f'(A_i) = g_i H_i g_i^{-1}$. The image of $A_{\mathcal{C}}$ by f' is generated by these conjugates of the subgroups H_i. It acts on the tree T_Γ corresponding to Γ. A surface type vertex group S of Γ corresponds to a surface which does not lie in $L(f)$, so it

intersects $f'(A_{\mathcal{C}})$ at most in a boundary subgroup. Thus, in the action of $f'(A_{\mathcal{C}})$, the corresponding vertex has cyclic stabiliser, and if it is not trivial, it stabilises an adjacent edge. This edge is unique by 1-acylindricity of surface type vertices, so by collapsing all such edges, we see that

$$f'(A_{\mathcal{C}}) = g_1 H_1 g_1^{-1} * g_2 H_2 g_2^{-1} * \ldots * g_m H_m g_m^{-1}.$$

Let β be the map which restricts on $g_i H_i g_i^{-1}$ to conjugation by g_i^{-1}. The map f' sends intact vertex groups to non-abelian images by Lemma 8.12, hence so does $\beta \circ f'$ since β is an isomorphism between $g_1 H_1 g_1^{-1} * \ldots * g_m H_m g_m^{-1}$ and $H_1 * \ldots * H_m$. Similarly, by Lemma 8.13, f' preserves disjointness of conjugacy classes of edge groups of the graphs Λ_i, hence so does $\beta \circ f'$. Note that

$$\beta(f'(A_{\mathcal{C}})) = H_1 * \ldots * H_m = A'.$$

If $m \geq 1$, then $H_1 * \ldots * H_m$ is clearly non-cyclic. But if $m = 1$, the image of $A = A_1$ by f is contained in H_1, so H_1 is not abelian. Thus, by Lemma 8.10, we can find a map $\tau : R_{\mathcal{C}} \to \beta(f'(A_{\mathcal{C}}))$ such that the map $F = [(\beta \circ f') * \tau] \circ \rho_{\mathcal{C}}$ sends surface type vertex groups to non-abelian images.

$$\begin{tikzcd}
 & A_{\mathcal{C}} \arrow[r, "f'"] & f'(A_{\mathcal{C}}) \arrow[r, "\beta"] & A' \\
A \arrow[r, "\rho_{\mathcal{C}}"] & * & & \\
 & R_{\mathcal{C}} \arrow[rruu, dotted, "\exists\tau"'] & &
\end{tikzcd}$$

Moreover, it is easy to see that the map F sends each subgroup H_i isomorphically on itself.

Thus the restriction η of F to $A' = H_1 * \ldots * H_m$ is an isomorphism $A' \to A'$. Finally, the map $\eta^{-1} \circ F$ restricts to the identity on A', so it is a retraction r from A to A'. Moreover, it sends surface type vertex groups of Γ to non-abelian images. Now the graph of groups with surfaces Γ is bipartite between surface and non surface type vertices, its non surface type vertex groups are the groups H_i, which generate the subgroup A', and which form a free product in A. Thus (A, A', r) is a hyperbolic floor. This concludes the proof of Proposition 5.11.

8.6 Proof of Proposition 5.12

Let G be a torsion-free hyperbolic group, and let A be a retract of G which admits a JSJ-like decomposition Λ. Suppose G' is a subgroup of G containing A such that either G' is a free factor of G, or G' is a retract of G by a retraction $r : G \to G'$ which makes (G, G', r) a hyperbolic floor.

Denote by $r : G \to G'$ the retraction which is the trivial map on R if $G = G' * R$, and the retraction of the hyperbolic floor structure in the second case. Let Γ be the graph of group corresponding to the free product $G' * R$ in the first case, and the graph of groups decomposition associated to the hyperbolic floor structure in the second case.

Let $f : A \to G$ be the preretraction given by the hypotheses of Proposition 5.12. Choose a maximal essential set $\mathcal{C}$ of curves pinched by f on the surfaces of Λ. Let f' be such that $f = f' \circ \rho_{\mathcal{C}}$. Build the pinching of Λ by $\mathcal{C}$, and choose a pinching decomposition of $\rho_{\mathcal{C}}(A)$. Choose also a maximal system of essential curves $\mathcal{C}^+$ which gives an elliptic refinement Λ_i^+ for each Λ_i, with respect to f' and Γ.

Lemma 8.23: *The map $r \circ f'|_{A_{\mathcal{C}}}$ sends intact vertex groups of the Λ_i to non-abelian images.*

Proof. Let S be an intact vertex group of Λ_i. By Lemma 8.12, it is a surface type vertex, in particular it inherits a decomposition $\Delta(\Sigma, \mathcal{C}^+)$ from the refinement Λ_i^+.

If we are in the case where $G = G' * R$, the set of curves of $\mathcal{C}^+$ lying on the surface Σ corresponding to S is empty. Indeed, elements corresponding to curves of $\mathcal{C}^+$ are sent to edge groups of Γ by f', but edge groups of Γ are trivial and f' is non-pinching with respect to Λ_i so there can be no curves of $\mathcal{C}^+$ on Σ. Thus $f'(S)$ is elliptic in Γ. Since boundary subgroups of S are sent to non-trivial subgroups of a conjugate of A, $f'(S)$ lies in a conjugate of G', and by Lemma 8.12, it is non-abelian. Thus its image by r is non-abelian.

Let us now assume we are in the case where (G, G', r) is a hyperbolic floor. By Lemma 8.12, the image of at least one of the vertex groups S^+ of $\Delta(\Sigma, \mathcal{C}^+)$ has non-abelian image by f'. If $f'(S^+)$

lies in a non surface type vertex group of Γ, it lies in a conjugate of G', so its image by r is clearly non-abelian. Now note that f' sends edge groups of the graphs of groups Λ_i^+ into conjugates of G'. Thus if S^+ is sent by f' into one of the surface type vertex groups S_1 of T_Γ, its boundary elements must be sent to boundary elements of S_1, so by Lemma 3.10 $f'(S^+)$ is a subgroup of finite index of S_1. Now this means $r(f'(S^+))$ is a finite index subgroup of $r(S_1)$, which is not abelian by definition of a hyperbolic floor. Hence $r(f(S))$ is non-abelian. $\square$

$$\begin{tikzcd} & A_{\mathcal{C}} \arrow[r, "f'"] & f'(A_{\mathcal{C}}) \arrow[r, "r"] & G' \\ A \arrow[r, "\rho_{\mathcal{C}}"] & * & & \\ & R_{\mathcal{C}} \arrow[uurr, dotted, "\exists\tau"'] & & \end{tikzcd}$$

Let B_1 and B_2 be edge groups of the graphs of groups Λ_{j_1} and Λ_{j_2} whose conjugacy classes are disjoint in $\rho_{\mathcal{C}}(A)$. By Lemma 8.13, since A is a retract of G, the groups $f'(B_1)$ and $f'(B_2)$ have disjoint conjugacy classes. Now the image by f' of each B_i is the conjugate in G of an edge group C_i of Λ. The groups C_1 and C_2 lie in A, and have disjoint conjugacy classes in G. Since r is the identity on G', we see that $r(C_1) = r(f'(B_1))$ and $r(C_2) = r(f'(B_2))$ have disjoint conjugacy classes in G'. Thus, the map $r \circ f'$ also preserves disjointness of conjugacy classes of edge groups of the graphs Λ_i. Note that G' contains A, so that it isn't cyclic. Now we can apply Lemma 8.10 to $r \circ f'|_{A_{\mathcal{C}}}$, to get a map $\tau : R_{\mathcal{C}} \to G'$ such that the map $[(r \circ f'|_{\mathcal{C}}) * \tau] \circ \rho_{\mathcal{C}}$ sends surface type vertex groups of Λ on non-abelian images. It is easy to see that this map restricts to conjugation on each non surface type vertex group of Λ. This shows precisely that it is a preretraction $A \to G'$. If $\mathcal{C}$ is not empty, $\rho_{\mathcal{C}}$ is not injective, thus so is $[(r \circ f'|_{\mathcal{C}}) * \tau] \circ \rho_{\mathcal{C}}$. If $\mathcal{C}$ is empty, $[(r \circ f'|_{\mathcal{C}}) * \tau] \circ \rho_{\mathcal{C}}$ is just $r \circ f$ so it is also non-injective. This concludes the proof of Proposition 5.12.

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